Adaptive Grid Solution of Risk Sensitive Estimator Problems

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Abstract **- An on-going work proposing a novel method for numerical computation of risk sensitive state estimates for non-linear non-Gaussian problems is reported. The algorithm is based on point mass approximation also called the grid method and utilises a modified form of information state based recursive relation, proposed and proved as a theorem. The modified form is claimed to be more efficient for numerical evaluation of risk sensitive estimate, especially for aposteriori risk sensitive state estimation. Though grid based filters are known for low numerical efficiency, heuristics for adaptive choice of grid points has been proposed to alleviate the shortcoming. The performance of this filter is demonstrated with a linear Gaussian case. Salient features of this Adaptive Grid RSF is then contrasted against the recently proposed Risk Sensitive Filters using the particle approach..**

Keyword **- Adaptive Grid filter, Risk sensitive filter, Particle Filter**

I. INTRODUCTION

Risk Sensitive Estimators (RSE) have been known for well over a decade. RSE uses an exponential of accumulated cost and is believed to have increased robustness compared to risk neutral estimators and is closely linked with type of robust estimators [1]. Interest in RSEs has been renewed in [1], where a more general framework compared to the earlier publications [5,7,8,10,11] has been used. Essential general properties of RSE have been well compiled in [7]. In a companion paper [4] further insights have been presented.

For Linear Gaussian RSE problems, closed form recursive relations, called RSKF [4] are available [1,4,6,7]. The concept of Risk sensitive estimators (RSE) is applicable to linear as well as nonlinear problems. An EKF-like framework for RSE (which may be called as RSEKF) has been given in [9]. However, the limitations of EKF including smoothness requirement for the functions and noise Gaussianity restriction, are well known. Until recently when Risk Sensitive Particle filter [RSPF] was proposed [2], there had been no general method for solving RSE problems for non-Gaussian or non linear, non-smooth cases. The Adaptive Grid Risk Sensitive Filter (AGRSF) algorithm proposed here, is another candidate to fit the same situation i.e, non-linear, non-Gaussian risk sensitive state estimation.

We first present a necessary theorem with which recursive solution of general RSE may be deduced. The theorem has similarities with those previously published earlier [1,6,8], but a more intuitive inductive proof is provided. The particular form is amenable to a more efficient AGRSF implementation, compared to the previously available forms.

Though the recursive relations have been known for some time, the integration involved, in general, cannot be evaluated in closed form. Both RSPF and grid based methods [3] provide systematic methods for the necessary numerical integration and employ a point mass approximation to continuous probability densities.

The more popular Monte Carlo based methods, known as particle filter [2,3], employ random samples. In contrast, 'approximate' Grid based filter [3], a predecessor to particle filter employs deterministic grid points. Ordinary grid based methods suffer from many disadvantages like poor numerical efficiency and finite resolution. In the present work, some of these shortcomings are proposed to be mitigated by adaptation of grids.

The concept of grid adaptation had previously been attempted [11]. Some would also call particle filters as randomised adaptive grid filter. However, use of adaptive grid filter in the risk sensitive case is claimed to be novel requiring its own recursive formulae and own type of adaptation.

The RSE problem is stated in section II and the theorem giving the recursive relation is presented in section III. The Adaptive Grid RSE method is presented in section IV. The method is validated against Kalman Filter for Linear Gaussian Problem in section V. Discussion and conclusions are given in section VI.

II. RISK SENSITIVE ESTIMATION PROBLEM

Consider a general (non-linear) signal model consisting of equations for the state ($x_k \in R_n$) and measurement $y_k \in R^p$ with additive, uncorrelated noise $w_k \in R_n$, $y_k \in R^p$ of known statistics at the instance $k = \{0,1,2,3 \ldots n\}$.

$$x_{k+1} = f(x_k) + w_k \tag{1}$$

$$y_k = g(x_k) + v_k \tag{2}$$

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The vectors $f(x_k)$ and are general (without any assumption of smoothness) non-linear function of x_k and k. The initial state x_0 another random variable is uncorrelated with the noises mentioned above, but with known probability density distribution P_0.

A two parameter, risk sensitive squared error kernel type cost function at the time instance k may be defined as:

$$J_{RS}(\varsigma,k) = E[\exp(\mu_1 \sum_{i=0}^{k-1} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) + \mu_2 (x_k - \varsigma)^T (x_k - \varsigma))] \quad (3)$$

where $\hat{x}_i$s are the optimum estimated values of state variable for past steps $i \in \{0,1,2,3\ldots k\text{-}1\}$. The current optimum estimate $\hat{x}_k$ is obtained by finding the optimum value of, ζ which minimises

$$J_{RS}(\xi,k), \text{ i.e. } \hat{x}_k = \arg\min_{\varsigma} J_{RS}(\varsigma,k) \quad (4)$$

The constant parameters μ_1 and μ_2 are called risk sensitive parameters.

III. Recursive Solution Of The RSE

The recursive solution of RSE may be realized generally in a two-step process. First, a recursive relation of an information state [1,6] is formulated, which can be updated in each time step. In each time step, the optimal estimate is then obtained by extremizing the cost of another function involving the information state.

Theorem 1

The solution of the RSE problem may be obtained from the following recursive relations:

$$\hat{x}_{k+1|k+1} = \arg\min_{\varsigma} \int_{-\infty}^{+\infty} \exp[\mu_2 (x_{k+1} - \varsigma)^T (x_{k+1} - \varsigma)] \alpha_{k+1} dx_{k+1} \quad (5)$$

where

$$\alpha_{k+1} = p(y_{k+1} \mid x_{k+1}) \times \int_{-\infty}^{+\infty} \exp[\mu_1 (x_k - \hat{x}_k)^T (x_k - \hat{x}_k)] \alpha_k p(x_{k+1} \mid x_k) dx_k \quad (6)$$

Let the probability distributions of w_k, v_k be given respectively by $P_w(.)$, $P_v(.)$. The two probability density terms in equation (6) can then be expressed as

$$p(y_{k+1} \mid x_{k+1}) = p_v(y_{k+1} - g(x_{k+1}))$$
$$p(x_{k+1} \mid x_k) = p_w(x_{k+1} - f(x_k))$$

In Appendix 1, we provide an inductive method for deriving the expression for the information state. The induction gives a direct and intuitive derivation, without the use of Measure change, Radon-Nikodyme derivatives and Girsanov's theorem [1,6,12].

Notes: The relations, expressed in Theorem 1 are different from those obtained in [6]. Compared to the expression in [6], the expressions herein may be shown to be numerically more efficient in recursive update of information state and finding the optimal posterior estimate.

IV. The Adaptive Grid Formulation

The integrations given in relations (5) and (6), may not, in general be performed in closed form for non-linear or non Gaussian case. A convenient numerical technique is point mass approximation, which is used in grid based filters [2].

A. Representing probability densities as point mass on grid points:

The problem is to approximate at time step k, a probability distribution $p_k(x)$ defined everywhere in state space denoted by x. Let it be known apriori that outside the region G bounded by $x=X_0$ and $x=X_f$, the probability density is negligibly small.

The region of interest G, the state space is then subdivided into N grid points, (with not necessarily equal intervals). In particular, at time step k, let the nontrivial grid points (assumed to be located at the centroid of the hyper volume enclosed by intersecting adjacent grids) in state space be denoted as for x_k^j, j= 1,2,...N. These points are also called the support points.

In the simplest case, consider a single state variable. From prior knowledge, one can set the minimum and maximum values and divide the interval into N grids. The central point between grid line zero and grid line 1, may be called as the grid point 1. Similarly grid point j will be in the centre of grid line j-1 and j. The j^{th} hyper volume in this case would be the line segment between the j^{th} and $j\text{-}1^{th}$ grid lines.

In two dimensional case, state variables may be divided into N_1 and N_2 grid lines, creating $N=N_1 \times N_2$ grid points. The area bounded by the adjacent grid lines will be the corresponding grid hyper volume (surface). The grid points need to be numbered systematically. The idea of grid and support points is shown in fig.1.

For grid point x_k^j, let the probability mass be m_k^j and the corresponding grid hyper volume be h_k^j. The average probability density would be m_k^j/h_k^j. We also note that $\sum_{j=1}^{N} m_k^j = 1$. The probability density may be approximated by point mass as

$$p(x_k) \approx \sum_{i=1}^{N} m_k^i \delta(x_k - x_k^i)$$

Though the information state α_k is not a probability density, it does represent a density. It may therefore be approximated likewise. Support points for α_k are generally chosen to be the same as for $p(x_k)$. It may be seen easily that estimation is unaffected α_k if is multiplied by a constant factor. It is often advantageous to normalize the distribution of α_k.

B. The Grid based RSE algorithm:

The initial state variable is first discretized into grid and point mass as .

$$p_{0|0}(x_0) \approx \sum_{i=1}^{N} m_0^i \delta(x_0 - x_0^i)$$

The information state, likewise may be discretized as

$$\alpha_0(\varsigma) \approx \sum_{i=1}^{N} \alpha_0^i \delta(\varsigma - x_0^i)$$

The information state is updated from k-th to k+1-th step as:

$$\alpha_{k+1}^i(x_{k+1}^i) = p_v(y_{k+1} - g(x_{k+1}^i))$$
$$\times \sum_{j=1}^{N} \{\exp[\mu_1(x_k^i - \hat{x}_k)^T(x_k^i - \hat{x}_k)]\alpha_k^j(x_k^j) p_w(x_{k+1}^i - f(x_k^j))\} \quad (7)$$

The convolution-like summation expression comes from the theory of approximate grid based filter [3] and it requires the bulk of computational effort. Unlike the expression given in [1], the distribution $P_v(.)$ appears outside the summation and provides substantial computational economy.

The alpha masses are then normalized so that these sum into unity. The objective function to be minimised may then be approximated for any arbitrary point ζ as:

$$J_{RS}(\varsigma, k) \approx \sum_{j=1}^{N} \exp[\mu_2(x_k^j - \varsigma)^T(x_k^j - \varsigma)]\alpha_k^j$$

The risk sensitive state estimate is obtained by determining the optimising solution for the objective function. Note that, compared to [6], the objective function above is simpler and optimisation is easier. Note also the difference with [1]. Out of several possible methods for determining the optimum, the following has been used in the present work and is therefore recommended.

To obtain the optimum, solve for the zero value of the partial derivative of the objective function, i.e.,

$$\frac{\partial}{\partial \varsigma} J_{RS}(\varsigma, k) = 0 \Rightarrow \sum_{j=1}^{N} (x_k^j - \varsigma)\alpha_k^j \exp[\mu_2(x_k^j - \varsigma)^T$$
$$(x_k^j - \varsigma)]\alpha_k^j = 0$$

Standard techniques like Newton-Raphson method may then be employed.

C. The Adaptive Grid algorithm:

Compared to particle filters, the grid method tends to be computationally more expensive. One reason for the lack of numerical efficiency lies in the choice of the supports. If fixed support points are used (as is the case for ordinary grid based filters), either many such support points are redundant and contribute only to numerical load and not the accuracy of estimates or else, if too coarse a grid is chosen, estimation accuracy suffers. The adaptive grid method uses heuristics to choose a "good set of support points". By a good set, one means that there would be enough support points in core areas where a solution is expected and adequate points to represent the density in the fringe areas. For non-linear (also non-Gaussian) cases, the following heuristics have been adopted in the present work.

- An approximate value of $\hat{x}_{k+1} \approx \zeta_{k+1}$ is obtained, say by time update equation only or better by RSEKF [9].
- If RSEKF is used, approximate error covariance $\tilde{P}_{k+1}$ comes as a by-product. Else, approximate covariance is to be obtained by the time update consideration.
- The core area, defined above may be taken as the one-sigma (principal square root of $\tilde{P}_{k+1}$) boundary around ζ_{k+1}. The fringe area could be the three-sigma boundary, leaving out the core. (see fig.1).
- Approximately half of the support points are taken for the core area and the rest for the fringe.

The algorithm described above is called Adaptive Grid Risk Sensitive Filter (AGRSF).

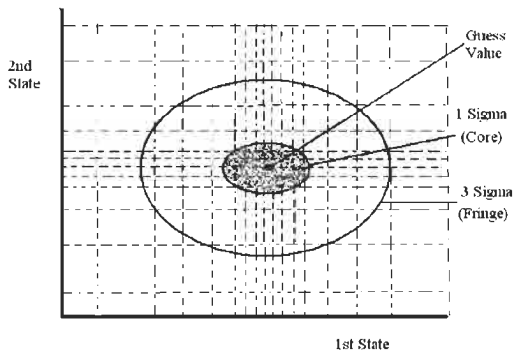

Fig.1. Idea of Grid and Supports

V. VALIDATION OF AGRSF

The AGRSF has been validated against Risk Sensitive Kalman Filter, mentioned before, for linear Gaussian case. Fig.2 shows a typical run with a first order system described by

$x_{k+1}=0.99x_k+bias+w_k, w_k \sim N(0,0.2), y_k=x_k+v_k,$
$v_k \sim N(0,1), x_0 \sim N(0,1),$ where $N(mean,sd)$ denotes normal distribution with mean and s.d. as indicated. The bias is a constant, set equal to 0.2 and is assumed to be unknown to the filter.

Fig.2 shows a typical run for 20 steps. It may be seen that the AG RSF estimate with 50 grids is virtually coincident with the ideal RS estimate obtained by RSKF. The improved estimation of risk sensitive filter compared to KF is also evident. The

tendency of convergence is shown in fig.3, where the absolute errors between AGRSF with different number of grid points and RSKF estimates have been plotted for the same example. In that figure the value against AGRSF indicate the number of grid points used in a particular MC run, example AGRSF-5 indicates that 5 grid points has been used. For a first order non linear case the AGRSKF estimate has been compared with the output of Risk sensitive particle filter[1] [2] as well as the same for RSEKF and all the three outputs are in reasonable agreement.

The trend as above has also been verified by Monte Carlo Run[1].

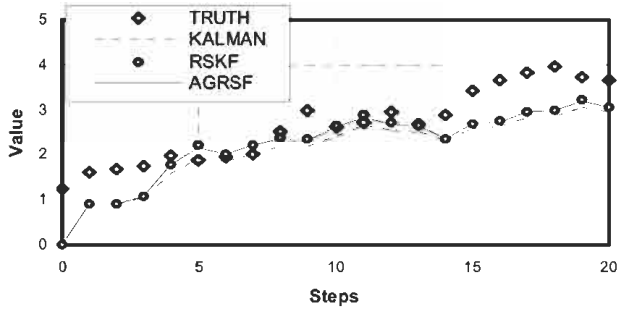

Fig.2. Comparison of AGRSF and KF for linear Systems

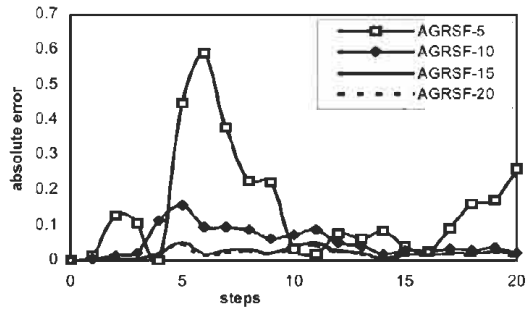

Fig.3. Convergence of AGRSF with number of grid points

VI. Discussions & Conclusion

Recursive relation for general Risk Sensitive Filters are derived through an alternative, information state based induction method. An algorithm for grid-based approximation of the recursive relation has been presented. Heuristics to improve the accuracy at a given computational budget using grid-adaptation has been suggested. The results have been compared with KF for linear Gaussian cases and with particle filter for a non-linear case. It is shown how the estimation accuracy improves with increased number of grid points.

Presently, the proposed method (AGRSF) and the Risk Sensitive Particle Filters (RSPF) [2] are the only methods reported for solving risk sensitive filtering problem for non-linear and/or non-Gaussian problems.

Compared to RSPF, the proposed method has been found to be numerically less efficient for higher order systems. In RSPF, one needs to choose appropriate proposal densities, for non-linear and/or non-Gaussian problems. Likewise, the Grid Adaptation policy, in AGRSF as of now, is evolving and only further work will tell whether a "Universal" adaptation policy can be found to cover a wide range of problems. With these known limitations AGRSF has been found to be a useful tool to gain insight into Risk Sensitive filtering and also for cross validation RSEKF and RSPF implementations.

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APPENDIXI: PROOF OF THEOREM-I

Starting from the chain rule, of conditional density [3], it is straightforward to show that

$$p(x_0, x_1, x_2, ...x_n \mid y_0, y_1, y_2, ...y_n)$$
$$= p(y_n \mid x_n) p(x_n \mid x_{n-1}) p(x_0, x_1, x_2, ...x_{n-1} \mid y_0, y_1, y_2, ...y_{n-1})$$

We derive the relations for the posterior estimation case first. In this context, for brevity $\hat{x}_k$ will mean $\hat{x}_{k|k}$. The first posterior estimate $\hat{x}_1$ is obtained as follows: from (5)

Define $\alpha_{0|0} = p(x_{0|0}) = p(x_0 \mid y_0), \quad \hat{x}_0 = mean(p(x_{0|0}))$

$$\hat{x}_{1|1} = \arg\min_{\varsigma} J_{RS}(\varsigma, 1) =$$

$$= \arg\min_{\varsigma} E[\exp(\mu_1 (x_0 - \hat{x}_0)^T (x_0 - \hat{x}_0) + \mu_2 (x_1 - \varsigma)^T (x_1 - \varsigma)]$$

$$= \arg\min_{\varsigma} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[\mu_1 (x_0 - \hat{x}_0)^T (x_0 - \hat{x}_0)] \exp[\mu_2 (x_1 - \varsigma)^T (x_1 - \varsigma)]$$
$$\times p(y_1 \mid x_1) p(x_1 \mid x_0) p(x_0 \mid y_0) dx_0 dx_1$$

by chain rule. Now let

$$\alpha_1 = p(y_1 \mid x_1) \int_{-\infty}^{+\infty} \exp[\mu_1 (x_0 - \hat{x}_0)^T (x_0 - \hat{x}_0)] \alpha_0 p(x_1 \mid x_0) dx_0$$

then, $$\hat{x}_{1|1} = \arg\min_{\varsigma} \int_{-\infty}^{+\infty} \exp[\mu_2 (x_1 - \varsigma)^T (x_1 - \varsigma)] \alpha_1 dx_1$$

similarly $\hat{x}_2$ can be obtained as

$$\hat{x}_{2|2} = \arg\min_{\varsigma} \int_{\infty}^{+\infty} \exp[\mu_2 (x_2 - \varsigma)^T (x_2 - \varsigma)] \alpha_2 dx_2$$

where

$$\alpha_2 = p(y_2 \mid x_2) \int_{-\infty}^{+\infty} \exp[\mu_1 (x_1 - \hat{x}_1)^T (x_1 - \hat{x}_1)] \alpha_1 p(x_2 \mid x_1) dx_1$$

Proceeding in this manner. by the method of induction one can conclude that

$$\hat{x}_{k+1|k+1} = \arg\min_{\varsigma} \int_{-\infty}^{+\infty} \exp[\mu_2 (x_{k+1} - \varsigma)^T (x_{k+1} - \varsigma)] \alpha_{k+1} dx_{k+1}$$

where

$$\alpha_{k+1} = p(y_{k+1} \mid x_{k+1})$$
$$\times \int_{-\infty}^{-\infty} \exp[\mu_1 (x_k - \hat{x}_k)^T (x_k - \hat{x}_k)] \alpha_k p(x_{k+1} \mid x_k) dx_k$$

For prior estimate (also called delayed measurement estimate [7]) may be obtained similarly by starting from $\hat{x}_{0|-1}$.

$p_0'(x_0) = p(x_0 \mid y_0) = p(x_0) p(y_0 \mid x_0)$, such that

$$\hat{x}_{0|0} = \arg\min_{\varsigma} E[\exp(\mu_2 (x_{0|-1} - \varsigma)^T (x_{0|-1} - \varsigma))]$$

$$= \arg\min_{\varsigma} \int_{-\infty}^{-\infty} \exp[\mu_2 (x_{0|-1} - \varsigma)^T (x_{0|-1} - \varsigma)] p(x_{0|-1}) dx_{0|-1}$$

$$= \arg\min_{\varsigma} \int_{-\infty}^{+\infty} \exp[\mu_2 (x_{0|-1} - \varsigma)^T (x_{0|-1} - \varsigma)] \alpha_0 dx_{0|-1}$$

Notes: 1) The recursive relation needs to be initialised for the information state. (2) The estimation equation is specifically applicable for posterior estimation where the estimation is performed after receiving the measurement. Similar relations can be obtained for prior estimates, also known as delayed measurement [7] as is used in [1].