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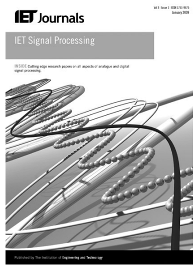

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Cubature quadrature Kalman filter

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Abstract: In this correspondence, the authors develop a novel method based on spherical radial cubature and Gauss–Laguerre quadrature rule for non-linear state estimation problems. The proposed filter, referred as cubature quadrature Kalman filter (CQKF) would be able to overcome inherent disadvantages associated with the earlier reported cubature Kalman filter (CKF). The theory and formulation of CQKF has been presented. Using two well-known non-linear examples, the superior performance of CQKF has been demonstrated. Owing to computational efficiency (compared to the particle and grid-based filter) and enhanced accuracy compared to the extended Kalman filter and the CKF, the developed algorithm may find place in on-board real life applications.

1 Introduction

Considerable amount of researches have been made to formulate efficient sub-optimal filtering algorithm for non-linear non-Gaussian systems because researchers have abandoned the idea of obtaining analytical solution. As a consequence, in the Bayesian framework, several numerical filtering techniques have evolved. In one approach, the intractable integrals encountered during non-linear and/or non-Gaussian estimation have been approximated as Gaussian, and characterised through mean and covariance using deterministic sample points, alternatively known as quadrature points, and weights associated with them. The non-linear filters, namely unscented Kalman filter [1, 2], Gauss–Hermite filter (GHF) [3], central difference filter [4] etc. belong to this category.

In another approach, researchers construct the probability density function (pdf) which embodies all available statistical information using points in state space (known as particles) and corresponding weights. The method is popularly known as particle filter (PF) [5] or sequential sampling resampling algorithm. Moreover, adaptive grid points are also been used to evaluate the pdf of states [6]. Although the PF and adaptive grid [7] filters can be implemented with desired accuracy, the computational cost is very high and suffer from 'the curse of dimensionality' problem.

In this paper, which is an extended version of earlier conference paper [8], we develop a novel algorithm where the multivariate moment integral has been computed numerically using the third-order cubature rule and multiple Gauss–Laguerre quadrature points. As it is concluded in [9] that more than third degree of cubature rule yields no improvement in performance, we investigate the incorporation of higher order quadrature points only.

To develop the new non-linear estimation technique, we have to compute a multidimensional integral in the form of 'non-linear function × Gaussian pdf'. In this paper, we solved the intractable integrals by using multidimensional cubature and single dimensional quadrature rule of integration. We propose the inclusion of multiple quadrature points for enhanced accuracy. The method of integration developed here has been used to solve the intractable integrals. As the integration method is more accurate, the developed estimation algorithm based on the proposed method of integration is more accurate than conventional methods of non-linear filtering.

The proposed estimator is named as cubature quadrature Kalman filter (CQKF). The CQKF would be more generalised form of cubature Kalman filter (CKF), which has been recently proposed by Arasaratnam and Haykin [9, 10]. The CKF algorithm evaluates the intractable integrals using third degree of spherical radial cubature rule. Under single Gauss–Laguerre quadrature point evaluation, the CQKF's performance coincides with the CKF.

The accuracy of proposed estimator depends on the order of the quadrature rule. The higher the number of quadrature points the better the accuracy would be. The particle and grid point requirements for the PF and grid filters [7] increase exponentially with the dimension of state. Also to implement n' order of GHF [11] for an n-dimensional system n'^n number of quadrature points are required. So the above mentioned filters have the 'curse of dimensionality' problem because the support point requirement increases exponentially with the dimension of the problem. However, for the proposed cubature quadrature (CQ) filter of order n', $2nn'$ number of support points and weights are required. So the number of points necessary for CQKF increases linearly with the dimension of the state. Thus the proposed filter does not suffer from the curse of dimensionality. Unlike the extended Kalman filter (EKF) [12], the proposed filter is derivative free, which may be considered as an added advantage.

The performance of the CQKF has been tested experimentally on two non-linear state estimation problems. In the first problem, the proposed method is used to estimate the state of a first-order severely non-linear system. The second problem deals with the estimation of the states of a Lorenz system under chaotic behaviour. The results of all the experiments demonstrate the improvement of performance with CQKF over conventional non-linear filters.

The rest of this paper is organised as follows. The next section presents the Bayesian framework of filtering and estimation. This is followed by CQ evaluation of multi-dimensional integrals. Section 4 elaborates the algorithm of CQKF. Simulation results are discussed in Section 5 and concluding remarks are in Section 6.

2 Filtering under Bayesian framework

Let us consider a non-linear plant described by the state and measurement equations as follows

$$x_{k+1} = \phi(x_k) + \eta_k \tag{1}$$

$$y_k = \gamma(x_k) + v_k \tag{2}$$

where $x_k \in R^n$ denotes the state of the system, $y_k \in R^p$ is the measurement at the instant k where $k = \{0, 1, 2, 3, \ldots, N\}$, $\phi(x_k)$ and $\gamma(x_k)$ are known non-linear functions of x_k and k. The process noise $\eta_k \in R^n$ and measurement noise $v_k \in R^p$ are assumed to be uncorrelated and normally distributed with covariance Q_k and R_k, respectively.

In the Bayesian estimation paradigm, the state x_k is to be estimated recursively at time k considering measurement data, $y_{1:k}$, up to time k. The prior probability density can be given by Chapman–Kolmogorov equation [7]

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1}) p(x_{k-1}|y_{1:k-1})\, \mathrm{d}x_{k-1} \tag{3}$$

The above equation is known as the time update equation. The computation of the posterior density function is done via Bayes' rule

$$p(x_k|y_{1:k}) = \frac{p(y_k|x_k) p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})} \tag{4}$$

where the normalising constant

$$p(y_k|y_{1:k-1}) = \int p(y_k|x_k) p(x_k|y_{1:k-1})\, dx_k \tag{5}$$

For linear Gaussian systems, the posterior and prior densities remain Gaussian in nature and the estimated values can be obtained optimally by the celebrated Kalman filter. Although for non-linear systems, the density functions are no longer Gaussian in nature, many times it is approximated as Gaussian and subsequently mean and covariance of prior as well as posterior density functions are evaluated.

From the above discussions, it is clear that to obtain estimated states, the integrals (3) and (4) need to be evaluated. As the integrals cannot be solved analytically, several numerical integration techniques [13] have been proposed. Further, the accuracy of the estimation depends on the accuracy of the approximate evaluation of the integrals. In the CQKF formulation, the nth order intractable integral is expressed in the spherical polar co ordinate system. The $(n-1)$th order integral is solved using spherical cubature rule and remaining first-order integral is approximated with higher order Gauss–Laguerre quadrature method.

2.1 Time update

The 'prior estimate' is the mean of the prior pdf obtained from the 'time update' equation. So

$$\begin{aligned}\hat{x}_{k|k-1} &= E[x_k|y_{1:k-1}] \\ &= E[(\phi(x_{k-1}) + \eta_k)|y_{1:k-1}] \\ &= E[\phi(x_{k-1})|y_{1:k-1}]\end{aligned}$$

or

$$\begin{aligned}\hat{x}_{k|k-1} &= \int \phi(x_{k-1}) p(x_{k-1}|y_{1:k-1}) dx_{k-1} \\ &= \int \phi(x_{k-1}) \aleph(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}) dx_{k-1}\end{aligned}$$

$$\begin{aligned}P_{k|k-1} &= E\left[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^{\mathrm{T}}|y_{1:k-1}\right] \\ &= \int \phi(x_{k-1}) \phi^{\mathrm{T}}(x_{k-1}) \aleph(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}) dx_{k-1} \\ &\quad - \hat{x}_{k|k-1}\hat{x}^{\mathrm{T}}_{k|k-1} + Q_k\end{aligned}$$

2.2 Measurement update

The measurement update under the approximation of the Gaussian posteriori pdf is given by

$$p(y_k|y_{1:k-1}) = \aleph\left(y_k; \hat{y}_{k|k-1}, P_{yy,k|k-1}\right)$$

where

$$\hat{y}_{k|k-1} = \int \gamma(x_k) \aleph(x_k; \hat{x}_{k|k-1}, P_{k|k-1})\, \mathrm{d}x_k$$

$$\begin{aligned}P_{yy,k|k-1} &= \int \gamma(x_k) \gamma^{\mathrm{T}}(x_k) \aleph(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \mathrm{d}x_k \\ &\quad - \hat{y}_{k|k-1}\hat{y}^{\mathrm{T}}_{k|k-1} + R_k\end{aligned}$$

The cross-covariance is given by

$$P_{xy,k|k-1} = \int x_k \gamma^{\mathrm{T}}(x_k) \aleph(x_k; \hat{x}_{k|k-1}, P_{k|k-1})\, \mathrm{d}x_k - \hat{x}_{k|k-1}\hat{y}^{\mathrm{T}}_{k|k-1}$$

On the receipt of new measurement, y_k, the posterior density

$$p(x_k|y_{1:k}) = \aleph(x_k; \hat{x}_{k|k}, P_{k|k})$$

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where

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\left(y_k - \hat{y}_{k|k-1}\right)$$
$$P_{k|k} = P_{k|k-1} - K_k P_{yy,k|k-1} K_k^{\mathrm{T}}$$
$$K_k = P_{xy,k|k-1} P_{yy,k|k-1}^{-1}$$

3 CQ evaluation of multidimensional integrals

3.1 Approach

The intractable integrals described in the previous section are solved numerically by finding a set of points and weights associated with them. The support points and the corresponding weights approximate the integrals by weighted sum of the functions evaluated at those points. To compute the integrals in (3) and (4), we decompose them into a surface integral and a line integral. It is transformed in such a way that the surface integral is calculated over the unit hyper-sphere of dimension n. The reason behind such a transformation is that the surface integral over a hyper-sphere could be calculated using spherical cubature rule and the line integral could be evaluated using the Gauss–Laguerre quadrature rule of integration. In simple words, we propose a method to compute the desired numerical integrals in a more accurate way. When the proposed method of integration is embedded in the Bayesian framework of filtering, a new estimation technique named CQKF has evolved. The proposed CQKF is more accurate compared to the EKF because the EKF uses first-order linearisation to approximately calculate the mean and covariance of the non-Gaussian pdf, whereas the CQKF uses cubature rule and Gauss–Laguerre quadrature points to determine the mean and covariance of the posterior pdf. The moment calculation is more accurate in the CQKF compared to the EKF. The accuracy of proposed method is expected to be more compared to the CKF because of the incorporation of higher order Gauss–Laguerre quadrature rule for calculation of intractable integrals. The cubature rule in [9] is described in a roundabout way. Here a more straight forward and easy to follow derivation is provided in the form of Theorem 1.

3.2 Cubature rule

Theorem 1: For an arbitrary function $f(x)$, $X \in R^n$ the integral

$$I(f) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} \int_{R^n} f(x) \mathrm{e}^{-(1/2)(X-\mu)^{\mathrm{T}}\Sigma^{-1}(X-\mu)}\, \mathrm{d}X$$

can be expressed in spherical coordinate system as

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \int_{r=0}^{\infty} \int_{U_n} \left[f(\boldsymbol{C}rZ + \mu)\, \mathrm{d}s(Z)\right] r^{n-1} \mathrm{e}^{-r^2/2}\, \mathrm{d}r \tag{6}$$

where $X = \boldsymbol{C}rZ + \mu$, $\boldsymbol{C}$ is the Cholesky decomposition of covariance matrix, $\boldsymbol{\Sigma}$, $\|Z\| = 1$, μ is the mean of Gaussian distribution and U_n is the surface of unit hyper-sphere.

Proof: Let us transform the integral $I(f)$ to a spherical coordinate system [14]. Let $X = \boldsymbol{C}Y + \mu$, $Y \in R^n$, where $\boldsymbol{\Sigma} = \boldsymbol{C}\boldsymbol{C}^{\mathrm{T}}$, that is, $\boldsymbol{C}$ is the Cholesky decomposition of $\boldsymbol{\Sigma}$. Then $(X-\mu)^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(X-\mu) = Y^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}^{-\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{C}Y = Y^{\mathrm{T}}Y$ and $\mathrm{d}X = |\boldsymbol{C}|\mathrm{d}Y = \sqrt{|\boldsymbol{\Sigma}|}\, \mathrm{d}Y$. So the desired integral

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \int_{R^n} f(\boldsymbol{C}Y + \mu) \mathrm{e}^{-(1/2)Y^{\mathrm{T}}Y}\, \mathrm{d}Y \tag{7}$$

Now let $Y = rZ$, with $\|Z\| = \sqrt{Z^{\mathrm{T}}Z} = 1$, $Y^{\mathrm{T}}Y = Z^{\mathrm{T}}rrZ = r^2$. The elementary volume of hyper-sphere at n-dimensional space is $\mathrm{d}Y = r^{n-1}\,\mathrm{d}r\,\mathrm{d}s(Z)$, where $\mathrm{d}s(.)$ is the area element on U_n. U_n is the surface of hyper-sphere defined by $U_n = \{Z \in R^n | ZZ^{\mathrm{T}} = 1\}$; $r \in [0, \infty)$. Hence

$$\begin{aligned} I(f) &= \frac{1}{\sqrt{(2\pi)^n}} \int_{r=0}^{\infty} \int_{U_n} f(\boldsymbol{C}rZ + \mu) \mathrm{e}^{-r^2/2} r^{n-1}\, \mathrm{d}r\, \mathrm{d}s(Z) \\ &= \frac{1}{\sqrt{(2\pi)^n}} \int_{r=0}^{\infty} \int_{Un} \left[f(\boldsymbol{C}rZ + \mu)\, \mathrm{d}s(Z)\right] r^{n-1} \mathrm{e}^{-r^2/2}\, \mathrm{d}r \end{aligned} \tag{8}$$

To compute the integral, $I(f)$, as described above, first we need to compute

$$\int_{U_n} f(\boldsymbol{C}rZ + \mu)\, \mathrm{d}s(Z) \tag{9}$$

The integral (9) can be approximately calculated by third degree fully symmetric spherical radial cubature rule. If we consider zero mean unity variance, (9) can be approximated as [9]

$$\int_{U_n} f(rZ)\, \mathrm{d}s(Z) \simeq \frac{2\sqrt{\pi^n}}{2n\Gamma(n/2)} \sum_{i=1}^{2n} f[ru_i] \tag{10}$$

where $[u_i]$ ($i = 1, 2, \ldots, 2n$) are the cubature points located at the intersections of the unit hyper-sphere and its axes. For example, in single dimension, the two cubature points will be on $+1$ and -1. For two dimensions, the four cubature points will be on $(+1, 0)$, $(-1, 0)$, $(0, +1)$ and $(0, -1)$. For Gaussian distribution with non-zero mean and non-unity covariance, the cubature points will be located at $(C[u_i] + \mu)$. As it is concluded in [9] that more than third degree of cubature rule yields no improvement in performance, only the inclusion of higher order of quadrature points has been investigated here.

3.3 Gauss–Laguerre quadrature rule

Any integral of a function $f(\cdot)$ in the form of

$$\int_{\lambda=0}^{\infty} f(\lambda) \lambda^{\alpha} \mathrm{e}^{-\lambda}\, \mathrm{d}\lambda \tag{11}$$

can be approximately evaluated using quadrature points and weights associated with them. The error associated with the approximate evaluation of the integral depends on the number of quadrature points. The quadrature points can be determined from the roots of the n' order of

Chebyshev–Laguerre polynomial equation [15, 16].

$$L_{n'}^{\alpha}(\lambda) = (-1)^{n'} \lambda^{-\alpha} e^{\lambda} \frac{d^{n'}}{d\lambda^{n'}} \lambda^{\alpha+n'} e^{-\lambda} = 0 \qquad (12)$$

Let the quadrature points be $\lambda_{i'}$. The weights can be determined as

$$A_{i'} = \frac{n'!\Gamma(\alpha + n' + 1)}{\lambda_{i'}\left[\dot{L}_{n'}^{\alpha}(\lambda_{i'})\right]^2}$$

So the integral (11) can be written approximately using the quadrature rule as

$$\int_{\lambda=0}^{\infty} f(\lambda)\lambda^{\alpha} e^{-\lambda}\, d\lambda \simeq \sum_{i'=1}^{n'} A_{i'} f(\lambda_{i'})$$

3.4 CQ rule

Combining (8) and (10) we obtain

$$I(f) = \frac{1}{\sqrt{(2\pi)^n}} \times \frac{2\sqrt{\pi^n}}{2n\Gamma(n/2)} \int_{r=0}^{\infty} \left(\sum_{i=1}^{2n} f[ru_i] \right) r^{n-1} e^{-r^2/2}\, dr \qquad (13)$$

Now to integrate the rest of the term, we use the Gauss–Laguerre quadrature formula described above. To cast the integration in that form of (11), let us assume $t = r^2/2$. With this transformation (13) becomes

$$I(f) = \frac{1}{2n\Gamma(n/2)} \int_{t=0}^{\infty} \left(\sum_{i=1}^{2n} f\left[\sqrt{2t}u_i\right] \right) t^{(n/2-1)} e^{-t}\, dt \qquad (14)$$

Now the integration $\int_{t=0}^{\infty} f(t) t^{(n/2-1)} e^{-t}\, dt$ is approximated using multiple quadrature points with $\alpha = n/2 - 1$. As per earlier discussion, the accuracy of the estimator depends on the order of quadrature rule. For i' number of quadrature points denoted as $\lambda_{i'}$ the integral (14) becomes

$$I(f) = \frac{1}{2n\Gamma(n/2)} \times \left[\sum_{i=1}^{2n} \sum_{i'=1}^{n'} A_{i'} f\left(\sqrt{2\lambda_{i'}}\right)[u_i] \right]$$

For a n dimension of state space problem solved with third-order spherical cubature rule and n' order-Gauss–Laguerre quadrature points, total $2nn'$ points and associated weights need to be calculated. These points are named as CQ points in this paper.

3.5 Calculation of CQ points

The steps for calculating support points and associated weights are as follows:

- Find the cubature points $[u_i]_{(i=1,\ 2,\ldots,n)}$, located at the intersection of the unit hyper-sphere and its axes.
- Solve the n' order Chebyshev–Laguerre polynomial (12) for $\alpha = (n/2 - 1)$ to obtain the quadrature points ($\lambda_{i'}$).

$$L_{n'}^{\alpha}(\lambda) = \lambda^{n'} - \frac{n'}{1!}(n' + \alpha)\lambda^{n'-1} + \frac{n'(n'-1)}{2!}(n' + \alpha) \times (n' + \alpha - 1)\lambda^{n'-2} - \cdots = 0$$

- Find the CQ points as $\xi_j = \sqrt{2\lambda_{i'}}[u_i]$ and their corresponding weights as

$$w_j = \frac{1}{2n\Gamma(n/2)}(A_{i'}) = \frac{1}{2n\Gamma(n/2)} \frac{n'!\Gamma(\alpha + n' + 1)}{\lambda_{i'}\left[\dot{L}_{n'}^{\alpha}(\lambda_{i'})\right]^2}$$

for $i = 1, 2, \ldots, 2n$, $i' = 1, 2, \ldots, n'$ and $j = 1, 2, \ldots, 2nn'$.

Illustrations:

(a) Single dimensional case ($n = 1$)

First-order quadrature ($n' = 1$): For first-order quadrature, the two CQ points are $\{1 \ \ -1\}$ and their corresponding weights are $\{0.5\ 0.5\}$.

Second-order quadrature ($n' = 2$): For second-order quadrature, the four CQ points are $\{-2.3344 \ \ -0.7420 \ \ 0.7420 \ \ 2.3344\}$ and their corresponding weights are $\{0.0459\ 0.4541\ 0.4541\ 0.0459\}$.

(b) Two-dimensional (2D) case ($n = 2$)

First-order quadrature ($n' = 1$): In 2D state space with first-order quadrature, the four CQ points are

$$\begin{Bmatrix} 1.414 & 0 & -1.414 & 0 \\ 0 & 1.414 & 0 & -1.414 \end{Bmatrix}$$

and all have the same weights 0.25.

Second-order quadrature ($n' = 2$): For two dimensions and second-order quadrature, the eight CQ points and their weights are (see expression at the bottom of the page) as shown in Fig. 1. The above discussion illustrates that sum of all weights are unity. So the quadrature points and associated weights represent the discrete pdf.

4 CQKF algorithm

The algorithm of the proposed CQKF could be summarised as follows:

Step i Filter initialisation

$$\begin{Bmatrix} 2.6131 & 1.0824 & 0 & 0 & -2.6131 & -1.0824 & 0 & 0 \\ 0 & 0 & 2.6131 & 1.0824 & 0 & 0 & -2.6131 & -1.0824 \end{Bmatrix}$$
$$\text{and} \quad \{0.0366 \ \ 0.2134 \ \ 0.0366 \ \ 0.2134 \ \ 0.0366 \ \ 0.2134 \ \ 0.0366 \ \ 0.2134\}$$

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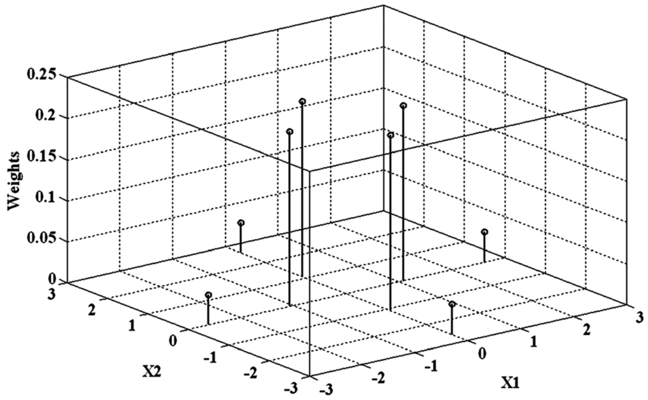

Fig. 1 *Plot of weights against CQ points for 2D system and second-order quadrature*

- Initialise the filter with $\hat{x}_{0|0}$ and $P_{0|0}$.
- Calculate the CQ points, ξ_j, and their corresponding weights $w_j (j = 1, 2, \ldots, 2nn')$.

Step ii Predictor step

- Perform the Cholesky decomposition of posterior error covariance

$$P_{k|k} = S_{k|k} S_{k|k}^{\mathrm{T}}$$

- Evaluate the CQ points

$$\chi_{j,k|k} = S_{k|k}\xi_j + \hat{x}_{k|k}$$

- Update the CQ points

$$\chi_{j,k+1|k} = \phi(\chi_{j,k|k})$$

- Compute the time updated mean and covariance

$$\hat{x}_{k+1|k} = \sum_{j=1}^{2nn'} w_j \chi_{j,k+1|k}$$

$$P_{k+1|k} = \sum_{j=1}^{2nn'} w_j [\chi_{j,k+1|k} - \hat{x}_{k+1|k}][\chi_{j,k+1|k} - \hat{x}_{k+1|k}]^{\mathrm{T}} + Q_k$$

Step iii Corrector step or measurement update

- Perform the Cholesky decomposition of prior error covariance

$$P_{k+1|k} = S_{k+1|k} S_{k+1|k}^{\mathrm{T}}$$

- Evaluate the CQ points

$$\chi_{j,k+1|k} = S_{k+1|k}\xi_j + \hat{x}_{k+1|k}$$

where $j = 1, 2, \ldots, 2nn'$.

- Find the predicted measurements at each CQ points

$$Y_{j,k+1|k} = \gamma\left(\chi_{j,k+1|k}\right)$$

- Estimate the predicted measurement

$$\hat{y}_{k+1} = \sum_{j=1}^{2nn'} w_j Y_{j,k+1|k}$$

- Calculate the covariances

$$P_{y_{k+1}y_{k+1}} = \sum_{j=1}^{2nn'} w_j \left[Y_{j,k+1|k} - \hat{y}_{k+1}\right]\left[Y_{j,k+1|k} - \hat{y}_{k+1}\right]^{\mathrm{T}} + R_k$$

$$P_{x_{k+1}y_{k+1}} = \sum_{j=1}^{2nn'} w_j \left[\chi_{j,k+1|k} - \hat{x}_{k+1|k}\right]\left[Y_{j,k+1|k} - \hat{y}_{k+1}\right]^{\mathrm{T}}$$

- Calculate Kalman gain

$$K_{k+1} = P_{x_{k+1}y_{k+1}} P_{y_{k+1}y_{k+1}}^{-1}$$

- Compute the posterior state values

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - \hat{y}_{k+1}\right)$$

- The posterior error covariance matrix is given by

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1} P_{y_{k+1}y_{k+1}} K_{k+1}^{\mathrm{T}}$$

Note 1: Compared to the EKF, the proposed filter is derivative free, that is, to implement it, neither a Jacobian nor Hessian matrix need to be calculated. This may be considered as an added advantage from the computational point of view.

Note 2: The accuracy of the filter depends on the order of Gauss–Laguerre quadrature. The higher the order, the more accurate the estimator would be.

Note 3: As discussed elsewhere, the CQKF is computationally efficient compared to the PF and grid-based filter because instead of generating large number of particles or grid points in the state space few support points are required to be generated, updated and predicted. The proposed method is also computationally more efficient than the GHF because to implement the n'th order-GHF for an n-dimensional system, n'^n number of quadrature points are necessary whereas to implement the nth order CQKF (CQKF-n) $2n'n$ number of support points are necessary. However, the computational cost of CQKF is slightly higher than EKF and CKF.

Note 4: For a fixed order of Gauss–Laguerre quadrature the CQ points as well as their weights can be calculated and stored off line.

Note 5: For the unscented or sigma point Kalman filter the sigma point, located at the centre is the most significant and carries highest weight, where as in CQKF there is no CQ point at the origin. It is concluded in [9] that the cubature approach is 'more principled in mathematical terms' than sigma point approach. The same argument is true for the proposed CQKF.

5 Simulation results

In this section, we present simulation results obtained by using the CQKF of two non-linear state estimation problems. Although the first problem is single dimensional in nature, it is difficult to track the states and extensively used in literature to compare the accuracy of a newly proposed algorithm with existing ones. The second problem deals with the popular Lorenz system in the discrete domain. Using the examples, we demonstrate the superiority of proposed algorithm compared with CKF and EKF [17].

Example 5.1: Single dimensional process: The plant model, inspired by the authors [4, 6], has one unstable and two stable equilibrium points and given by

$$x_{k+1} = \phi(x_k) + \eta_k$$

where

$$\phi(x) = x + \Delta t 5x(1 - x^2), \quad \eta_k \sim \aleph(0, b^2\Delta t)$$

The measurement model is

$$y_k = \gamma(x_k) + v_k$$

where

$$\gamma(x) = \Delta t x(1 - 0.5x), \quad v_k \sim \aleph(0, d^2\Delta t)$$

The following values are used: $\Delta t = 0.01$ s, $x_0 = -0.2$, $\hat{x}_{0|0} = 0.8$, $P_{0|0} = 2$, $b = 0.5$ and $d = 0.1$. We have considered the time span from 0 to 0.8 s. The system has three equilibrium points of which, the one at the origin is unstable and the other two are at ± 1 and stable. In the absence of any bias, the system settles around either of the two stable equilibrium points. The problem becomes challenging because moderate estimation error forces the estimate to settle down at the wrong equilibrium point, leading to a track loss situation.

For the single dimensional case, the surface integral becomes zero order which could be considered as the surface of a hyper-sphere in 1D. A hyper-sphere in 1D is a pair of points sometimes called a zero sphere and surface is zero dimensional. The remaining line integral is approximately evaluated using Gauss–Laguerre quadrature points. With the increase in the order of quadrature points, the evaluation of the integral becomes more accurate. Hence, with the higher order quadrature points the performance enhancement for a scalar system is expected.

Fig. 2 shows truth and estimated values obtained from EKF, CKF and CQKF (with second-order quadrature) for a single representative run. It has been observed that the EKF losses track while the other two filters do not. The performance of the proposed filter has been compared with the CKF in Fig. 3 using root-mean-square error (RMSE) obtained from 1000 Monte Carlo (MC) runs. It has been seen that the CQKF with second-order Gauss–Laguerre quadrature gives substantial better result than the CKF as formulated in [9]. For 1000 MC runs, the RMSE of CKF settles nearly on 0.5, whereas CQKF settles at 0.25.

The performance of the filters is also compared in terms of fail count. The fail count is defined as the number of cases where estimation error at 0.8 s is more than unity out of

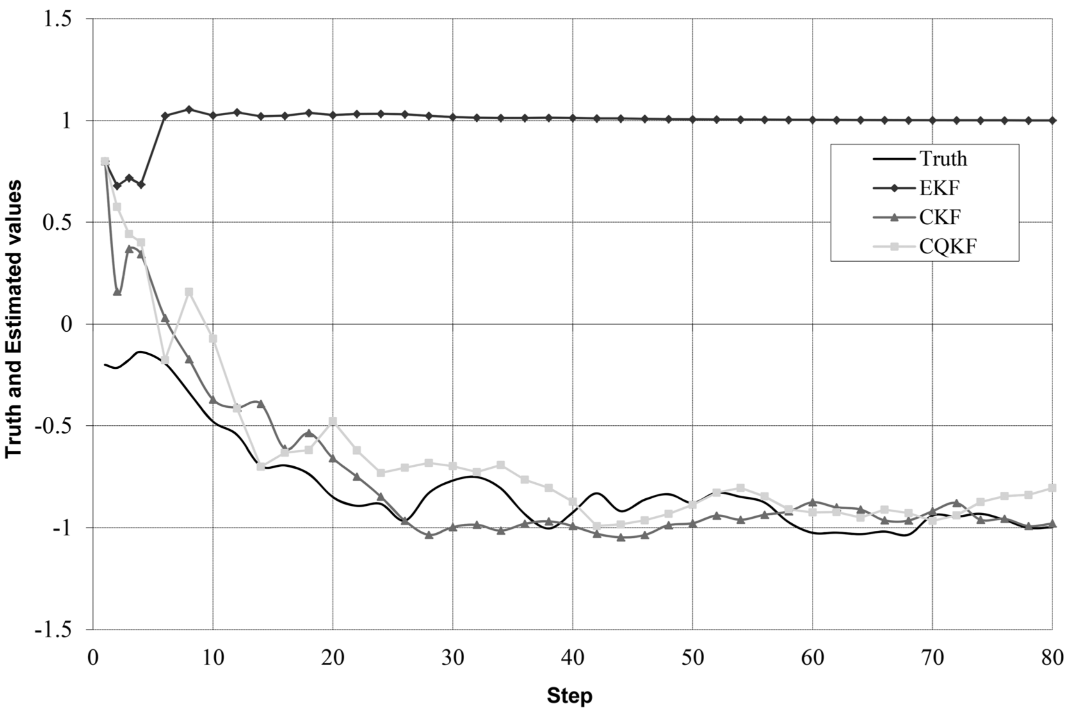

Fig. 2 *CKF and CQKF for single representative run*

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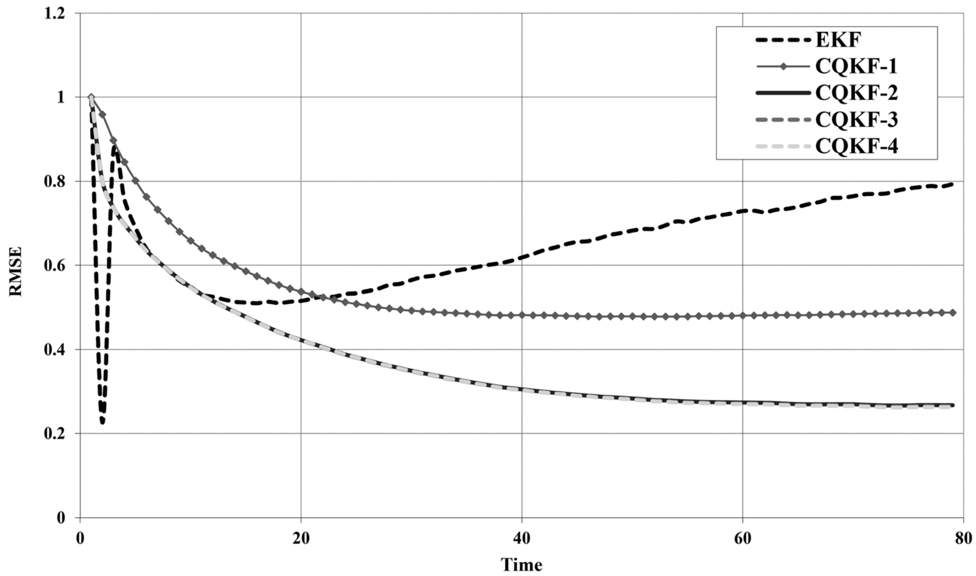

Fig. 3 *RMSE plot of CKF and CQKF for 100 MC runs*

100 MC runs. The percentage fail counts of different filters are summarised in Table 1. In Table 1, CQKF-x indicates the proposed CQKF filter with xth order Gauss–Laguerre quadrature points. The CQKF-1 represents the CQ filter with first-order Gauss–Laguerre quadrature point which is same as the CKF. The numbers indicate the improvement of estimation accuracy with the proposed filter in comparison to both standard EKF and more recent CKF [9].

Example 5.2: Lorenz system: Inspired from earlier works [4, 18], in this example, we consider the discrete time Lorenz attractor, one of the classic icons in non-linear dynamics. The attractor is available in the literature from 1963 mainly because of the work of meteorologist Edward Lorenz. Although higher dimensional Lorenz attractors may exist [19, 20] in real life problems, here we consider the 3D system. The process equation is given by

$$x_{k+1} = \phi(x_k) + b\eta_k$$

where

$$\phi(x) = x + \Delta t f(x), \quad \eta_k \sim \aleph(0, \Delta t)$$

The measurement equation is expressed as

$$y_k = \gamma(x_k) + dv_k$$

where

$$\gamma(x) = \Delta \mathrm{th}(x), \quad v_k \sim \aleph(0, \Delta t)$$

Table 1 Percentage fail count of different filters

Filter	Fail count, %
EKF	22
CKF/CQKF-1	6
CQKF-2	2.2
CQKF-3	2.1
CQKF-4	2.1

Here $x_k = [x_{1,k} \quad x_{2,k} \quad x_{3,k}] \in \mathbb{R}^3$ is state variable. The parameters $b \in \mathbb{R}^3$, and $d \in \mathbb{R}$ are constant whose values are taken as $b = [0 \quad 0 \quad 0.5]^{\mathrm{T}}$ and $d = 0.065$. The functions $f(x)$ and $h(x)$ are given by

$$f(x) = [\alpha(-x_1 + x_2) \quad \beta x_1 - x_2 - x_1x_3 \quad -\gamma x_3 + x_1x_2]^{\mathrm{T}}$$

and

$$h(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

The three parameters, α (called as Prandtl number), β (called as Rayleigh number) and γ have great impact on the system. The system has three unstable equilibrium points and for $\alpha \neq 0$, and $\gamma(\beta-1) \geq 0$ they are at $[0 \quad 0 \quad 0]^{\mathrm{T}}$, $[\sqrt{\gamma(\beta-1)} \quad \sqrt{\gamma(\beta-1)} \quad (\beta-1)]^{\mathrm{T}}$ and $[-\sqrt{\gamma(\beta-1)} \quad -\sqrt{\gamma(\beta-1)} \quad (\beta-1)]^{\mathrm{T}}$. We chose the system with classical parameter values: $\alpha = 10$, $\gamma = 8/3$ and $\beta = 28$, for which almost all points in phase space tend to a strange attractor [20].

The initial truth value of state is taken as $x_0 = [-0.2 \quad -0.3 \quad -0.5]^{\mathrm{T}}$. We simulate during the time interval of 0–4 s with the sampling time $\Delta t = 0.01$ s. The initial posterior estimate is assumed as $\hat{x}_{0|0} = [1.35 \quad -3 \quad 6]^{\mathrm{T}}$ with initial error covariance $P_{0|0} = 0.35I_3$.

Figs. 4, 5 show the RMSE plot of the states for 100 MC runs obtained by EKF, CKF and CQKF. For the third state, no appreciable improvement has been observed with the proposed filter (Fig. 6). However, for the other two states the RMSE of the CQKF is considerably small compared to the CKF and the EKF. From the figures, we conclude that the CQKF shows substantial improved performance compared to the CKF.

The relative computational times of the EKF and the CQKF (with different order Gauss–Laguerre quadrature points) have been tabulated in Table 2. Table 2 shows increase in computational cost with the increase in order of CQKF.

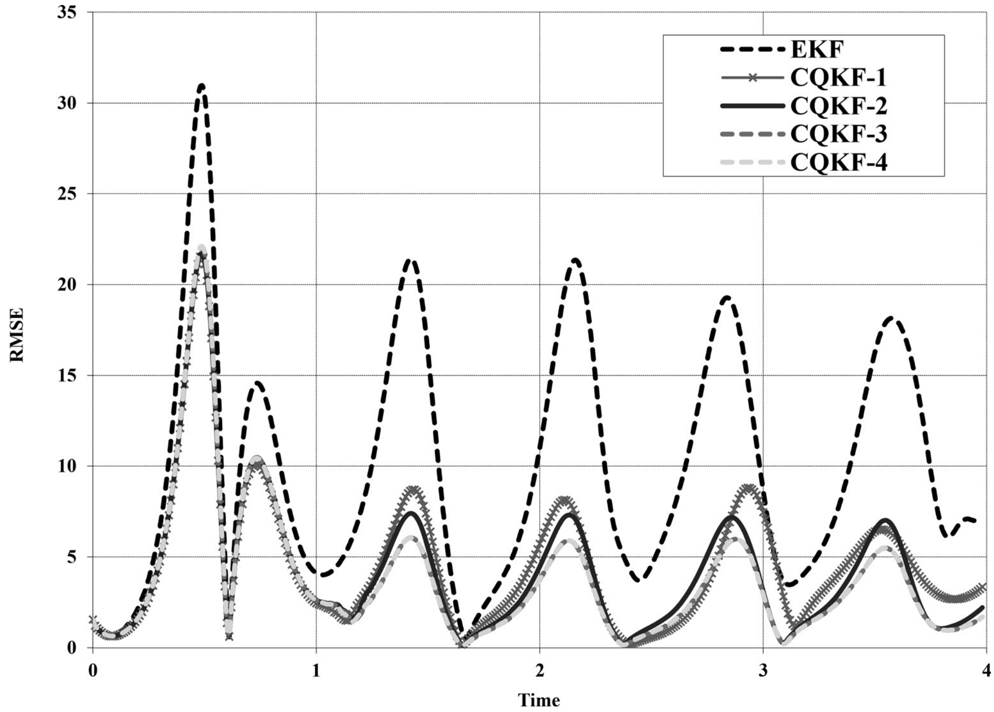

Fig. 4 *RMSE plot of first state variable using EKF, CKF and CQKF for 100 MC runs*

6 Discussions and conclusion

In this paper, for non-linear systems, we propose a novel algorithm based on numerical evaluation of intractable integral using the spherical radial cubature rule and multiple Gauss–Laguerre quadrature points. The developed CQKF would be more generalised form of CKF and under single Gauss–Laguerre quadrature point evaluation, it coincides with the CKF. The superiority of proposed method compared to the CKF has been demonstrated with the help of two well known examples. Owing to enhanced accuracy and computational efficiency (compared with particle and grid-based filters), the proposed filter may become a promising candidate for non-linear real-life applications.

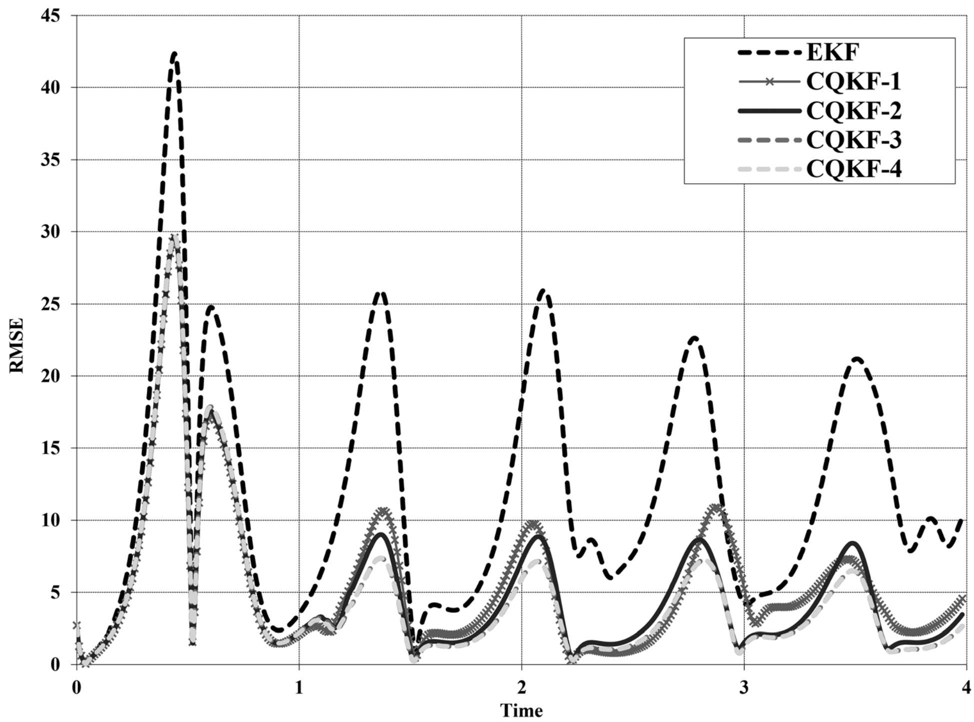

Fig. 5 *RMSE plot of second state variable using EKF, CKF and CQKF for 100 MC runs*

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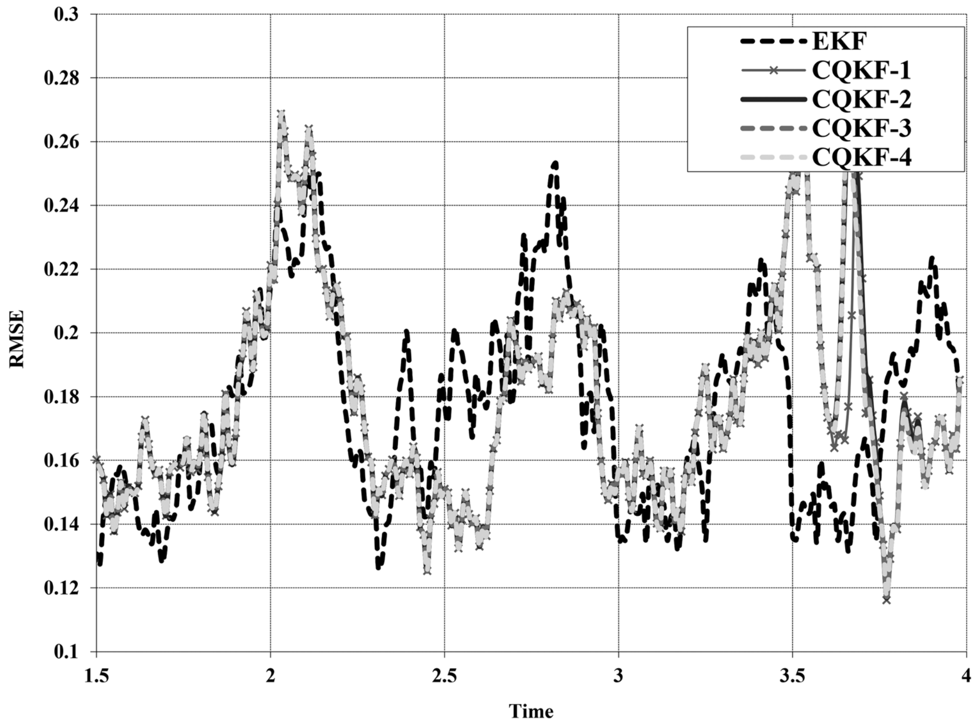

Fig. 6 *RMSE plot of third state variable using EKF, CKF and CQKF for 100 MC runs*

Table 2 Relative computational cost

Estimator	Relative computational time
EKF	1
CKF/CQKF-1	5.1289
CQKF-2	9.1687
CQKF-3	12.997
CQKF-4	16.983

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