Bearing Only Tracking Using Gauss-Hermite Filter

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***Abstract*—In this paper, performance of Gauss-Hermite filter (GHF) in bearing only tracking problem has been compared with that of extended Kalman filter (EKF) and unscented Kalman filter (UKF) in terms of estimation accuracy, probability of track-loss and computational efficiency. The performance improvement of the GHF with increase in quadrature points and enhanced robustness compared to EKF and UKF with respect to large initial uncertainty has been reported. It has been concluded that without introducing substantial computational burden, GHF with three or more quadrature points exhibits better performance compared to UKF and EKF.**

***Index Terms*—Bearing only tracking, Kalman Filter, Unscented Kalman filter, Gauss-Hermite filter**

I. Introduction

Bearing only tracking (BOT) has been an important field of study and has attracted the attention of many researchers [1], [2], [3], [4], [5] in the last few decades. BOT is used in many practical military and civil applications including under water weapon systems, infrared seeker based tracking, sonar based robotic navigation and TV camera or stereo microphone based people tracking to name a few. For weapon guidance system, BOT allows use of passive tracking sensors with the consequent tactical advantages. Further non-linearity and observability problem associated with BOT make the problem challenging and attractive to researchers.

Target tracking is generally carried out using seekers or sonars for aerospace and naval applications respectively. From the sensor either only bearing angle information or both bearing and range information are available. BOT becomes extremely important when range information is unavailable or the available information is ambiguous having unacceptable level of noise.

The literature on bearing only target tracking is rich and has been briefly described in the next section. The most recent version of it where the moving target is tracked from a moving platform, has been published by Lin *et al.* [6]. The authors have presented a comparative study of three nonlinear filters namely extended Kalman filter (EKF), pseudo measurement filter, and particle filter for a simple BOT problem where both target and platform are moving with some average velocity disturbed with random noise. The same problem has been adopted by Sadhu *et al.* [7] where relative merits of UKF compared to EKF and its variances have been established. Unlike real-life problems which are three dimensional in nature, both the papers [6], [7] considered single dimensional motion of target as well as tracking platform. In this work, the above mentioned simple kinematics has been considered to acquire more insight about Gauss-Hermite filtering technique.

Until recently, the extended Kalman filter (EKF) has been the natural choice of any designer to solve the nonlinear target tracking problems. The EKF uses local linearization technique to approximately calculate the mean and covariance of non Gaussian probability density function. Due to such crude approximation, the filter looses track if the nonlinearity or uncertainty in the system is high.

Recently more advanced algorithms are available where the intractable integrals, arise during estimation has been approximately solved using quadratures points. The nonlinear filters, namely unscented Kalman filter (UKF) [8], Gauss-Hermite filter (GHF) [9], central difference filter (CDF) [9], etc belong to this category. Among them UKF has received considerable amount of appreciation from the research community. The same is not true for GHF except for a few notable publications [10], [11].

In this paper, performance of GHF in BOT problem with respect to estimation accuracy, track-loss count and robustness against large initial condition has been investigated and compared with EKF and UKF. Also the performance improvement of GHF with increase in quadrature points is reported. Further, computational time necessary to realize the filters has been calculated and compared with conventional estimators.

II. Problem formulation

The earliest version of this particular BOT problem occurs in [12], where no platform noise is considered. Later Lin *et al.* [6] and Sadhu *et al.* [7], [13] re-formulated the problem incorporating random noise with platform motion. A brief overview of the problem is being presented in this section.

Bearing only tracking requires either (i) multiple tracking stations with known coordinates or (ii) a moving platform with known velocities, on which the tracker is mounted. The non-linear bearing only target-tracking problem used in [6] is of the second type. The problem may be visualized by an engagement scenario, where a low flying aircraft is tracking an ammunition train or an enemy ship. Though the numerical data provided in [6] is possibly more suitable for an under water scenario, we continue to use the low flying aircraft vocabulary for appreciation of the problem. The target is assumed to move in a straight line (considered as X-axis)

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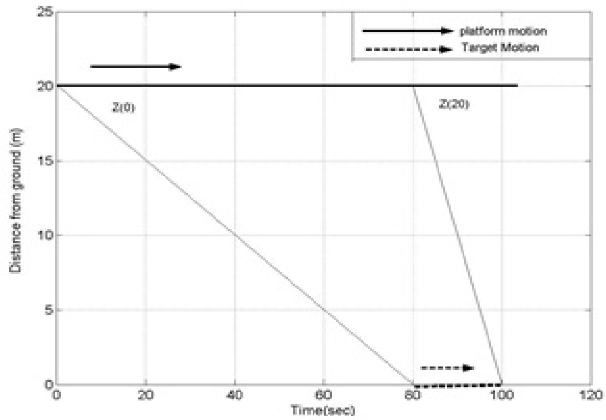

Fig. 1. Tracking platform kinematics

in the horizontal plane (fig-1), with a near constant velocity perturbed by the process noises. In order to track the target, the platform is also moving with a near constant velocity in the same vertical plane. Imperfections of platform motion are modelled by noises in X and Y directions. The average (neglecting the noisy imperfections) motions of the platform are known with negligible errors. The bearing measurement of the target, however, is also noisy. The measurement is processed to estimate the velocity and position of the target in an earth fixed coordinate system. The observer model is non-linear and both process and measurement noises are Gaussian. Note that the uncertainty in platform motion would appear as a measurement noise.

A. Process model

The BOT problem considered has two components, namely, the target kinematics and the tracking platform kinematics as shown in fig 1. The target is moving in X direction with the following discrete state space relations

$$X_{k+1} = F_k X_k + G_k w_k \tag{1}$$

$$X_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}, F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, G_k = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

where $x_{1,k}$ is the position along the X-axis in meters; $x_{2,k}$ is the velocity in m/sec; w_k is independent zero mean Gaussian white acceleration noise sequence with variance q. The sampling time is denoted as T, which has a nominal value of 1 sec. The (unknown) true initial condition and the known noise variance are assumed to be $X_0 = [80 \;\; 1]^T$ and $q = 0.01 m^2/s^4$ respectively. The target motion noise covariance matrix may be computed as

$$\begin{aligned} Q_k = G_k G_k^T &= [T^2/2 \;\; T]^T [T^2/2 \;\; T] q \\ &= \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix} q \end{aligned} \tag{2}$$

The tracking platform motion may be described by the following discrete time equations:

$$x_{p,k} = \bar{x}_{p,k} + \Delta x_{p,k} \qquad k = 0, 1, ..., n_{step} \tag{3}$$

$$y_{p,k} = \bar{y}_{p,k} + \Delta y_{p,k} \qquad k = 0, 1, ..., n_{step} \tag{4}$$

Where $\bar{x}_{p,k}$ and $\bar{y}_{p,k}$ are the (known) average platform position co-ordinates and $\Delta x_{p,k}$ and $\Delta y_{p,k}$ are the mutually independent zero mean Gaussian white noise sequences with variances $r_x = 1 \; m^2$ and $r_y = 1 \; m^2$, respectively. The mean positions of the platform are $\bar{x}_{p,k} = 4kT$ and $\bar{y}_{p,k} = 20$.

B. Measurement model

The measurement equation (in bearing coordinate) is given by

$$z_{m,k} = z_k + v_{s,k} \tag{5}$$

$$z_k = h[x_{p,k}, \; y_{p,k}, \; x_{1,k}] = tan^{-1} \frac{y_{p,k}}{x_{1,k} - x_{p,k}} \tag{6}$$

The bearing between the x axis and the line of sight (LOS) from the sensor to the target and $v_{s,k}$ is the zero mean Gaussian white measurement noise sequence with variance $(r_s = (3^\circ)^2)$, assumed to be independent of the sensor platform perturbations and sampling interval.

The random component of platform motion thus induces additional measurement error, which is non-additive and already embedded in equation(6). The above effect can be approximated as additive noise by expanding the non-linear measurement equations as

$$z_{m,k} = h[x_{p,k}, \; y_{p,k}, \; x_{1,k}] + v_{s,k} \approx h[\bar{x}_{p,k}, \; \bar{y}_{p,k}, \; x_{1,k}] + v_k \tag{7}$$

Where v_k is the equivalent additive measurement noise (with variance R_k) given approximately by small perturbation theory as

$$v_k \approx \frac{\bar{y}_p \Delta x_p + \{x_{1,k} - \bar{x}_{p,k}\} \Delta y_p}{[x_{1,k} - \bar{x}_{p,k}]^2 + \bar{y}_{p,k}^2} + v_{s,k} \tag{8}$$

R_k is calculated considering $\Delta x_{p,k}$, $\Delta y_{p,k}$ and v_k to be mutually independent

$$R_k = E[v_k^2] = \frac{\bar{y}_{p,k}^2 r_x + [x_{1,k} - \bar{x}_{p,k}]^2 r_y}{\{[x_{1,k} - \bar{x}_{p,k}]^2 + \bar{y}_{p,k}^2\}^2} + r_s \tag{9}$$

C. Filter Initialization

Traditionally, tracking filters are initialized from first two measurements. The latest bearing measurement defines the initial position estimate and the difference of two bearing measurements provides the estimation of initial velocity. The initial position estimate thus obtained may be shown to have a covariance of

$$P_{11,0} = r_x + \frac{r_y}{tan^2 z} + \frac{\bar{y}_p^2}{sin^4 z} r_s \tag{10}$$

The traditional way of initializing velocity estimation would create unduly large variances due to the large measurement error covariance in the present problem. Lin *et al.* [6] attempted to reduce such large errors by using prior knowledge about the target motion. As per [6] the initial position and velocity estimations are selected as $x_{1,0} = 80$ and $x_{2,0} = 0$ respectively and associated variance as $P_{22,0} = 1$. The off-diagonal terms $P_{12,0}$ and $P_{21,0}$ are taken as zero.

III. EKF AND UKF

Extended Kalman filter is used to estimate position and velocity of the target with the initial $x_{0|0}$, $P_{0|0}$ as described in the previous section. As the process model is linear only calculation of Jacobian matrix is necessary for measurement equation. The detailed algorithm of EKF is available in many standard books and literature, hence omitted here.

Unscented Kalman Filter, alternatively known as sigma point filter [8] or Julier-Uhlmann filter [9], uses sigma points or quadrature points and weights associated with them to capture the mean and covariance of posterior probability density function of states. The UKF captures posterior mean and covariance accurately up to the second order Taylor series expansion in contrast to EKF where only first order approximation is used. Hence UKF is expected to offer better result compared to EKF and for BOT the same has been reported in earlier literature [7]. For a nth order system to implement UKF generally $(2n+1)$ number of quadrature points need to be generated, propagated and updated at each time instance.

There are a few variants of UKF reported in the literature. Square root version of it has been proposed by Merwe *et al.* [14] for numerical stability and guaranteed positive definiteness of error covariance matrix. Another variant of UKF, named as scaled UKF (based on scaled unscented transformation) was proposed [15] where three parameters namely α, β, and κ were introduced to circumvent the numerical problems associated with ordinary UKF. In many problems the parameters are also used for fine tuning.

The ordinary UKF, introduced by Julier *et al.* [8] used Cholesky decomposition once to generate sigma points using posterior mean and covariance obtained from previous step. They are propagated and updated through process and measurement equations. In another version of UKF [14], prior and posterior sigma points are drawn from the prior and posterior mean and covariance. So Cholesky decomposition is done twice in contrast to former method where Cholesky decomposition is done once. Although conceptually both the variants are almost same but numerically they lead to different result. In this problem simple ordinary UKF as proposed by Julier [8] has been implemented. The algorithm has been omitted here because it is widely mentioned in earlier literature.

IV. GAUSS-HERMITE FILTER

Although Gauss-Hermite quadrature rule of integration was available in mathematics literature [16], [17] for more than fifty years ago, the same has been incorporated in signal processing very recently mainly due to work of Ito and Xiong [9]. The heart of GHF is the Gauss-Hermite quadrature rule of integration which makes it possible to approximately evaluate the intractable integrals encountered in nonlinear Bayesian filtering problem. A quadrature rule in general, approximately evaluates complicated integral by rewriting it as a product of non-negative weights and a function of quadrature points. The single dimensional Gauss-Hermite quadrature rule is given by

$$\int_{-\infty}^{\infty} f(x)\frac{1}{(2\pi)^{1/2}}e^{-x^2}dx = \sum_{i=1}^{N} f(q_i)w_i \tag{11}$$

where q_i and w_i represent N number of quadrature points and weights associated with them. To calculate the quadrature points, let us consider a symmetric tridiagonal matrix J with zero diagonal elements and $J_{i,i+1} = \sqrt{i/2}; 1 \le i \le N-1$. The quadrature points are located at $\sqrt{2x_i}$, where x_i are the eigenvalues of J [18]. The expression of quadrature points as described in [9] had an anomaly and has been corrected here. The weights w_i is the square of the first element of the i^{th} normalized vector. The extension of quadrature rules to multi-dimension problem could be done easily by using product rule. The n dimensional integral,

$$I_N = \int_{R_n} f(s)\frac{1}{(2\pi)^{n/2}}e^{-(1/2)|s|^2}ds \tag{12}$$

could be approximately evaluated as

$$I_N = \sum_{i_1=1}^{N} ... \sum_{i_n=1}^{N} f(q_{i_1}, q_{i_2}, ..., q_{i_n})w_{i_1}w_{i_2}...w_{i_n} \tag{13}$$

In order to evaluate I_N for nth order system, N^n number of quadrature points and weights are necessary. As an example for a second order system and three point GHF (from now on let us call as GHF-3) nine quadrature points and weights would be $\{(q_i, q_j)\}$ and $\{w_i w_j\}$ respectively for $i = 1, 2, 3$ and $j = 1, 2, 3$. From the discussion, it is clear that the number of quadrature points increase exponentially with the dimension of the system. So the GHF suffers from the *curse of dimensionality* problem. To a very limited extent this could be overcome by ignoring the quadrature points on the diagonals because the weights associated with them are very small hence they contribute negligibly to the computation of the integral. In this paper, all the quadrature points are retained and none ignored.

In this particular BOT problem, the process equation is linear and the measurement equation is non-linear. This allows us to use the Kalman filter (KF) algorithm for time propagation step and the GHF algorithm during measurement update. To distinguish our algorithm with the standard GHF, we call the filter as KF-GHF where time update step is KF type. The main benefit associated with the KF-GHF would be the computational efficiency. The GHF algorithm is available in earlier literature [9]; KF-GHF algorithm along with generation of quadrature points and weights is provided in Appendix.

V. SIMULATION RESULTS

The problem described in section II has been solved using Matlab simulation. Results are taken for 10,000 Monte Carlo runs. Finally we use ten such batches for studying performances of the different tracking filters. To ensure different random numbers at each step, each batch of Monte Carlo simulation has been re-seeded.

Figure 2 shows the Root mean square errors (RMSEs) of position of 100 Monte Carlo runs obtained from EKF, UKF,

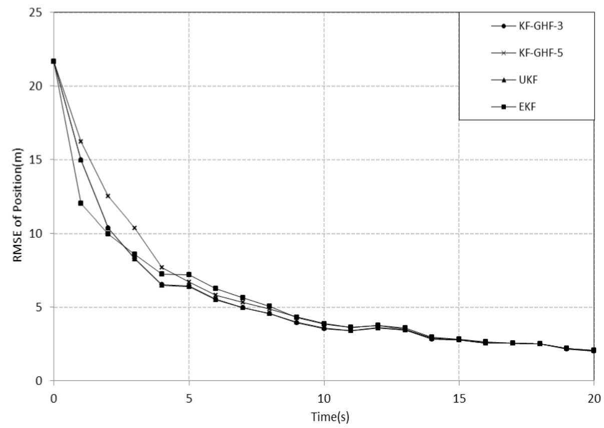

Fig. 2. RMSE of position for 100 MC runs

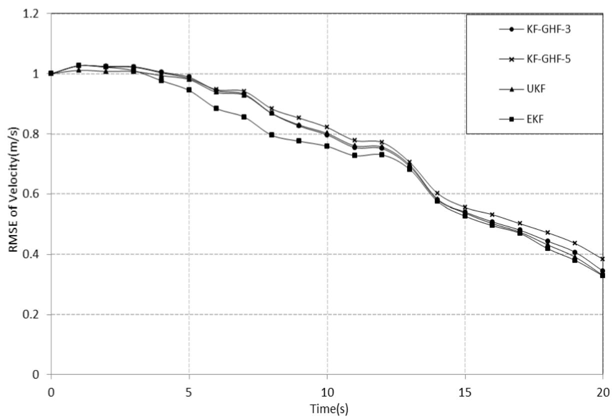

Fig. 3. RMSE of velocity for 100 MC runs

third order KF-GHF (we call it KF-GHF-3) and KF-GHF-5 excluding track-loss cases (the criteria of track-loss will be discussed in the next subsection). Figure 3 shows similar results for velocity of the target. The RMSEs of both position and velocity excluding track-loss cases are more or less same for all the filters. Further we explored other parameters such as track-loss count, computational efficiency and effect of initial uncertainty to assess the comparative performance of the filters.

A. Comparison of track-loss

The track-loss criteria is defined as $|x_{20} - \hat{x}_{20}| < X_{end}$, where $X_{end} = 15m$ and the subscript 20 denotes the 20th time step. The probability of track-loss was calculated over the population of one hundred thousand runs and has been displayed in table 1. The table 1 reveals that the track-loss of EKF is highest. In terms of track-loss three point GHF performs better than UKF. Compared to GHF-3, track-loss count decreases with GHF-5. However with higher order Gauss-Hermite filter no significant improvement of performance had been observed hence the results are omitted here.

TABLE I
% TRACK-LOSS AND RELATIVE COMPUTATIONAL TIME

Estimator	% Track-loss	Relative computational time
EKF	0.343	1
UKF	0.020	1.377
GHF-3	0.007	1.523
GHF-5	0.005	3.152
KF-GHF-3	0.007	1.317

B. Computational efficiency

Due to presence of Cholesky decomposition the computational complexity of both UKF and GHF is $O(n^3)$ whereas the computational complexity of EKF is $O(n^2)$. The relative execution times of all the filters have been tabulated in table 1. Computational time of GHF increases significantly with increase in order. Further KF-GHF algorithm significantly reduces computational cost compared to standard GHF filter. From table 1, we may claim that KF-GHF-3 outperforms UKF in terms of computational efficiency.

C. Effect of initial uncertainty

The count of track-loss is found to be deeply affected by the initial covariance of the filter. To study the effect of initial error covariance with the track-loss count, we set $P_0 = \mu \times P_{nom,0}$ and we vary the parameter μ from 1 to 10. Figure 4 shows the effect of μ on the percentage of fail count for EKF, UKF and KF-GHF-3. From figure 4 it can be seen that KF-GHF-3 shows better robustness against uncertainties in initial error covariance.

VI. DISCUSSIONS AND CONCLUSIONS

The track-loss criteria has been imposed only for position estimation because in all the simulations, velocity estimation converges with time. The percentage of track-loss for EKF reported here is much higher than earlier published papers [6], [7] due to tighter failure criteria. The performance evaluation of square root UKF [14] and GHF [19] which may reduce the computational time, remains under the scope of future work.

The findings, out of this work, may be summarized as follows:

- If track-loss cases are ignored the RMSEs obtained from EKF, UKF and KF-GHF are equivalent.
- KF-GHF-3 outperforms EKF and UKF in terms of track-loss count. Further GHF with order more that five does not provide any addition advantage, hence GHF with order higher than five is not recommended for this particular BOT problem.
- In terms of computational efficiency EKF is fastest followed by KF-GHF-3 and UKF. Also KF-GHF filter is significantly faster compare to ordinary GHF.
- The robustness study against initial uncertainty reveals that GHF is more robust than popular UKF and traditional EKF.

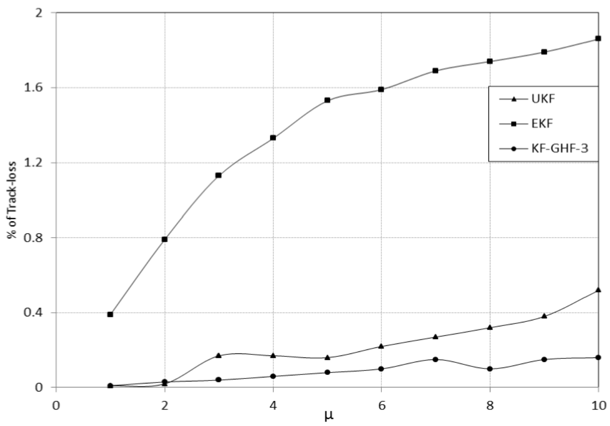

Fig. 4. Track-loss vs μ for EKF, UKF, and KF-GHF-3

In conclusion, our study indicates that GHF performs better than UKF in terms of robustness and track-loss count. Further if process or measurement equation is linear KF-GHF algorithm may be implemented which results in lower computational cost compared to UKF. Therefore implementation of KF-GHF is recommended for this particular BOT problem (linear target dynamics) and ordinary GHF for all other BOT problems, provided little increase in computational time is acceptable.

APPENDIX

System Description

The target dynamics in the problem is described by the process equation,

$$x_k = Fx_{k-1} + w_k$$

and by the measurement equation,

$$y_k = h(x_k) + v_k$$

x_k and y_k denote the states and the measurements of the system respectively at any instant k, where $k = \{0, 1, 2, 3, ..., N\}$. F is the system matrix and $h(x_k)$ is a known nonlinear function in x and k. The process and measurement noise w_k and v_k are white Gaussian with covariances Q_k and R_k respectively.

KF-GHF Algorithm

Step (i) Filter initialization

- Initialize with $\hat{x}_{0|0}$, and $P_{0|0}$
- compute quadrature points by $q_i = \sqrt{2}x_i$, where x_i are the eigenvalues of the symmetric tri-diagonal matrix J, elements of which are given by $J_{i,i+1} = \sqrt{i/2}$, $J_{i,i} = 0$.
- Calculate the respective weights w_i of the quadrature points q_i, where w_i is equal to the square of the first element of the i^{th} normalized eigenvector of J.

Step (ii) Predictor step

- Compute the predicted mean

$$x_{k|k-1} = Fx_{k-1|k-1}$$

- Compute the predicted covariance

$$P_{k|k-1} = FP_{k-1|k-1}F^T + Q_k$$

Step (iii) Corrector step or measurement update

- Perform Cholesky factorization of $P_{k|k-1}$ as

$$P_{k|k-1} = S^T S$$

- Modify the quadrature points as

$$\chi_i = S^T q_i + \hat{x}_{k-1|k-1}$$

- Propagate the modified quadrature points through the observation equation

$$\xi_i = h(\chi_i)$$

- Compute measurement mean

$$z_k = \sum_{i=1}^{N} \xi_i w_i$$

- Calculate the covariances

$$P_{zz} = R + \sum_{i=1}^{N} (\xi_i - z_k)(\xi_i - z_k)^T w_i$$

$$P_{xz} = \sum_{i=1}^{N} (\chi_i - x_{k|k-1})(\xi_i - z_k)^T w_i$$

- Calculate Kalman gain

$$K_k = P_{xz} P_{zz}^{-1}$$

- Posterior estimate is given by

$$\hat{x}_{k|k} = x_{k|k-1} + K_k(y_k - z_k)$$

- Evaluate posterior covariance

$$P_{k|k} = P_{k|k-1} - K_k P_{xz}^T$$

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