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On Detectability and Observer Design for Rectangular Linear Descriptor Systems

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Abstract A new method is proposed to check the detectability for a class of rectangular linear time invariant descriptor systems. The method is based on the properties of restricted system equivalent, derived here from a given descriptor system by means of simple matrix theory. Equivalence between the detectability of a given descriptor system and that of a normal system is established. The proposed result is applied to design full- and reduced-order observers for the same class of descriptor systems. Some illustrative examples are provided.

Keywords Descriptor systems · Detectability · Observer design · Pole placement · LMI

1 Introduction

In the last few decades, descriptor systems (also known as singular systems, implicit systems, differential algebraic equations (DAEs)) have attracted the attention of many researchers in the field of control theory and its applications since such systems are general enough to provide a solid understanding of the inner dynamics of any physical system [1, 3, 6]. In this paper, we consider descriptor systems of the form

$$\begin{aligned} \tilde{E}\dot{x} &= \tilde{A}x + \tilde{B}u, \\ \tilde{y} &= \tilde{C}x, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, $\tilde{y} \in \mathbb{R}^p$ are the state vector, the input vector and the output vector, respectively. $\tilde{E} \in \mathbb{R}^{m\times n}$, $\tilde{A} \in \mathbb{R}^{m\times n}$, $\tilde{B} \in \mathbb{R}^{m\times k}$, and $\tilde{C} \in \mathbb{R}^{p\times n}$ are known constant matrices, and the rank$(\tilde{E}) = n_0 < \max\{m, n\}$. System (1) is called regular descriptor system if $m = n$ and $\exists\, \lambda \in \mathbb{C}$ such that $\det(\lambda\tilde{E}-\tilde{A}) \neq 0$. In the case matrices $\tilde{E}$, $\tilde{A}$ are square and $\tilde{E}$ is invertible, the system (1) is called normal system.

Many physical systems can be modeled as the system of differential algebraic equations and can be expressed in the form of the system (1), but it is not always necessary that the number of variables of interest and equations are same. Thus, we are here concerned with rectangular systems. Some real life applications are electrical circuits [20], chemical control processes [14], constrained mechanics [23] to name a few.

In this paper, we assume system (1) satisfies the following conditions:

$$\text{(H1)}\quad \operatorname{rank}\begin{bmatrix} \tilde{E} & \tilde{A} \\ 0 & \tilde{E} \\ 0 & \tilde{C} \end{bmatrix} = n + \operatorname{rank}(\tilde{E}),$$

$$\text{(H2)}\quad \operatorname{rank}\begin{bmatrix} \lambda\tilde{E} - \tilde{A} \\ \tilde{C} \end{bmatrix} = n \ \forall\ \lambda \ \in\ \bar{\mathbb{C}}^{+}.$$

where $\mathbb{C}$ represents the set of complex numbers. $\bar{\mathbb{C}}^{+} = \{s | s \in \mathbb{C}, Re(s) \geq 0\}$ is the closed right half complex plane.

In [4], authors have obtained that the assumptions (H1) and (H2) are sufficient to design an observer for

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systems of the form (1). Dai [2] has extended the concepts of controllability and observability for regular systems to rectangular cases. Adopting the same concepts, we define some of the following useful terms for the rectangular systems. For detailed technical meaning of the following terms for square descriptor systems, we refer to [6].

(1) System (1) is called I- (Impulse) observable if condition (H1) holds.
(2) System (1) is said to be detectable if condition (H2) holds.
(3) System (1) is called R- (Reachable) observable if condition (H2) holds $\forall\ \lambda\ \in\ \mathbb{C}$.
(4) System (1) is called completely detectable if both of the conditions (H1) and (H2) hold.

An observer is a mathematical realization which uses the input and output information of a given system and its output asymptotically approaches the true state values of the given system. A great deal of research has been conducted on the observer design problem for square descriptor systems [7, 9, 11, 15, 16, 19, 21, 24, 25, 26, 27], but results on rectangular systems are limited [4, 5, 12, 13, 17, 18]. Müller and Hou [12, 17, 18] have proved that the requirement of regularity is not necessary for designing observers for descriptor systems. Concepts of generalized Sylvester equation and generalized inverse have been used for the design of observers for descriptor systems without unknown inputs [4] and with unknown inputs [5]. Koenig and Mammar [13] have proposed a method to design full- and reduced-order proportional integral observers for rectangular descriptor systems.

It has been shown that the condition of detectability is necessary for the existence of any Luenberger type observer for any descriptor system [3]. Our observer design approach is inspired by the work of [4, 13] where complete detectability is used as sufficient condition for the existence of an observer for descriptor system (1). As compared to these articles, the proposed method is straightforward and simple to understand and implement. Our approach is based on the restricted system equivalent theory and does not require the concept of generalized Sylvester equation. In this paper, one full column rank matrix R is designed in such a way that its pre-multiplication to some matrices gives the design approach for full-order observer and its post-multiplication reveals the reduced-order observer design approach. The order of the proposed reduced-order observer is the dynamical order of the given descriptor system (1).

The rest of the paper is organized as follows. Section 2 establishes an equivalence between the detectability of given descriptor system and that of a normal system. Theorem 2 proves the detectability of a reduced-order matrix pair. Based on the results of Section 2, full- and reduced-order observers are designed in Section 3. To illustrate the derived results, numerical examples for detectability and observer design are given in Section 4. Finally, Section 5 concludes the paper.

2 Detectability Results

In this section the detectability result for the descriptor system (1) is established. Following the lines of [4], the singular value decomposition of matrix $\tilde{E}$ gives us a nonsingular matrix Q such that $Q\tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $Q\tilde{A} = \begin{bmatrix} A \\ A_1 \end{bmatrix}$, $Q\tilde{B} = \begin{bmatrix} B \\ B_1 \end{bmatrix}$, and the system (1) is restricted system equivalent to

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2}$$

where $E \in \mathbb{R}^{n_0 \times n}$ (full row rank), $A \in \mathbb{R}^{n_0 \times n}$, $B \in \mathbb{R}^{n_0 \times k}$, $A_1 \in \mathbb{R}^{(m-n_0)\times n}$, $B_1 \in \mathbb{R}^{(m-n_0)\times k}$, $y = \begin{bmatrix} -B_1 u \\ \tilde{y} \end{bmatrix} \in \mathbb{R}^t$, $C = \begin{bmatrix} A_1 \\ \tilde{C} \end{bmatrix} \in \mathbb{R}^{t \times n}$, and $t = m + p - n_0$. Moreover, if the system (1) is impulse observable, the system (2) is also impulse observable because

$$\operatorname{rank} \begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \tilde{E} & \tilde{A} \\ 0 & \tilde{E} \\ 0 & \tilde{C} \end{bmatrix} = n + n_0. \tag{3}$$

The above equation implies one remarkable fact that

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{4}$$

The detectability of the system (1) and (2) is also equivalent because (for details see [4])

$$\operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \lambda \tilde{E} - \tilde{A} \\ \tilde{C} \end{bmatrix} = n \ \forall\ \lambda\ \in\ \bar{\mathbb{C}}^+. \tag{5}$$

It is easy to show that if rectangular descriptor system (2) satisfies the equation (4), there exists a full column rank matrix $R \in \mathbb{R}^{n \times n_0}$ such that equivalent square system

$$\begin{aligned} RE\dot{x} &= RAx + RBu, \\ y &= Cx, \end{aligned} \tag{6}$$

satisfies the condition

$$\operatorname{rank} \begin{bmatrix} I_n - RE \\ C \end{bmatrix} = \operatorname{rank}(C), \tag{7}$$

where I_n is identity matrix of order n. For mathematical construction of matrix R, we refer to [10]. The aforesaid

R is not unique, but to compute this, one numerically reliable algorithm is given in the Appendix of this paper.

Before providing the main theorem, we define the detectability of a matrix matrix pair $(\mathcal{A}, \mathcal{C})$.

Definition 1 A matrix pair $(\mathcal{A}, \mathcal{C})$ is said to be detectable for some square matrix $\mathcal{A}$ of order n iff rank $\begin{bmatrix} \lambda I_n - \mathcal{A} \\ \mathcal{C} \end{bmatrix} = n \ \forall \ \lambda \ \in \ \bar{\mathbb{C}}^+$.

Theorem 1 *If the given system (1) is I-observable, then the following statements are equivalent.*

(i) Descriptor system (1) is detectable.
(ii) Matrix pair (RA, C) is detectable.

Proof It is obvious that equation (7) implies the existence of $M \in \mathbb{R}^{n \times t}$ such that

$$RE = I - MC. \tag{8}$$

Thus for any $\lambda \ \in \ \bar{\mathbb{C}}^+$, we have

$$\begin{aligned} \text{rank} \begin{bmatrix} \lambda \tilde{E} - \tilde{A} \\ \tilde{C} \end{bmatrix} &= \text{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \lambda RE - RA \\ C \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \lambda(I - MC) - RA \\ C \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} I & -\lambda M \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - RA \\ C \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \lambda I - RA \\ C \end{bmatrix}. \end{aligned}$$

Hence the Theorem is proved.

Remark 1 For a non-detectable descriptor system, in general, it is not easy to find particular λ such that condition (H2) does not hold. Under the assumption of I-observability, above Theorem presents a method to find such λ by finding the eigenvalues of matrix RA (see Example 1 of this paper).

Theorem 2 *If the given system (1) is completely detectable, then matrix pair (AR, CR) is detectable.*

Proof The equation (8) implies that

$$ER = I_{n_0} - R^+ MCR,$$

where R^+ is any left inverse of matrix R. Now, using equation (5), for any $\lambda \ \in \ \bar{\mathbb{C}}^+$

$$\text{rank} \begin{bmatrix} \lambda ER - AR \\ CR \end{bmatrix} = n_0.$$

Thus,

$$\begin{aligned} \text{rank} \begin{bmatrix} \lambda ER - AR \\ CR \end{bmatrix} &= \text{rank} \begin{bmatrix} \lambda(I - R^+ MCR) - AR \\ CR \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} I & -\lambda R^+ M \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - AR \\ CR \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \lambda I - AR \\ CR \end{bmatrix} = n_0. \end{aligned}$$

Hence the theorem is proved.

3 Observer Design

In this section, methods to design full- and reduced-order observers for system (1) are discussed.

3.1 Full-order Observer Design

The problem is to design matrices N, L, and M of compatible dimensions such that the following normal system becomes a full order state observer for system (1), *i.e.*, $\hat{x} \to x$ as $t \to \infty$:

$$\begin{aligned} \dot{z} &= Nz + RBu + Ly \\ \hat{x} &= z + My. \end{aligned} \tag{9}$$

where $z \in \mathbb{R}^n$, $N \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{n \times t}$ and $M \in \mathbb{R}^{n \times t}$, and R is the same matrix as discussed in the previous section.

From equations (9) and (6) the error vector

$$\begin{aligned} e &= x - \hat{x} \\ &= x - z - MCx \\ &= (I - MC)x - z \\ &= REx - z \end{aligned} \tag{10}$$

gives the dynamics:

$$\begin{aligned} \dot{e} &= RE\dot{x} - \dot{z} \\ &= RAx + RBu - (Nz + RBu + LCx) \\ &= (RA - LC)x - N(REx - e) \\ &= Ne + (RA - LC - NRE)x \\ &= Ne + (RA - LC - N + NMC)x \\ &= Ne. \end{aligned} \tag{11}$$

In the construction of equations (10) and (11), we have assumed the existence of matrices M, K, N, and L of compatible orders such that equation (8) is satisfied with following two more equations

$$N = RA - KC \tag{12}$$
$$K = L - NM. \tag{13}$$

The error dynamics (11) is asymptotic stable if and only if matrix N is stable, which means that eigenvalues of matrix N are in the open left half complex plane. Thus the problem of designing the state observer (9) is converted into the design of the matrices M, K, N, and L such that the equations (8), (12), and (13) are satisfied with the stability of matrix N. Now, for finding a matrix K such that N is stable, we have two approaches as explained in the following two subsections. Rest of the matrices can be calculated by the equations (8) and (13).

Thus, under the assumptions (H1) and (H2), we have explained the design procedure of the full-order observer for the system (1) that is summarized again in form of the following algorithm.

Algorithm 1. For Full-order observer for the descriptor system (1)

1. Reduce the given descriptor system (1) in the form of system (2) as described in Section 2.
2. Find matrix R for the matrix pair (E, C) by algorithm given in Appendix A.
3. Solve matrix equation (8) for unknown matrix M.
4. Find matrix K either by pole placement or LMI approach as described in Sections 3.1.1 and 3.1.2, respectively, such that $N = RA - KC$ is a stable matrix.
5. Using the equation (13), calculate $L = K + NM$.

3.1.1 Pole Placement Approach

Using the pole placement techniques for normal matrix pair (RA, C), we can find a matrix K such that the poles of matrix RA in the closed right half complex plane can be arbitrarily replaced in the open left half complex plane. Moreover, if given system is R-observable then all poles can be arbitrarily replaced. Pole placement technique is used to get desired rate of convergence of error dynamics (11). For this, assume the matrix $K \in \mathbb{R}^{n\times t}$ as a variable matrix with $n \times t$ unknowns elements. Then calculate the characteristic polynomials for corresponding desired eigenvalues of matrix N and for matrix $(RA - KC)$. After equating the coefficients of same order terms in these two polynomials, we achieve maximum $n+1$ equations to solve for unknown entries of matrix K. Solvability of this system of equations is guaranteed by the observability of matrix pair (RA, C). More details about pole placement techniques can be found in [22].

3.1.2 Linear Matrix Inequality (LMI) Approach

In this subsection, an alternative LMI approach is proposed to find matrix K in equation (12) such that N is a stable matrix by using the Lyapunov stability theory. Let the Lyapunov function be $V = e^T Pe$, where P is a positive definite matrix. Then using (11) and (12) we have

$$\begin{aligned}\dot{V} &= \dot{e}^T Pe + e^T P\dot{e}\\ &= e^T(RA - KC)^T Pe + e^T P(RA - KC)e\\ &= e^T(A^T R^T P + PRA - C^T\tilde{K}^T - \tilde{K}C)e\end{aligned}$$

where $\tilde{K} = PK$.

According to the Lyapunov stability theory, error dynamics (11) is asymptotically stable if there exists two matrices $\tilde{K}$ and P such that

$$P > 0 \tag{14}$$

and

$$A^T R^T P + PRA - C^T\tilde{K}^T - \tilde{K}C < 0. \tag{15}$$

Due to the detectability of matrix pair (RA, C), it is clear that problem (14)-(15) is feasible. Numerical solution for P and $\tilde{K}$ can be found by any LMI tool box. In this work, the MATLAB LMI tool box is used [8].

3.2 Reduced-order Observer Design

Theorem 3 *Let the given system (1) be completely detectable. Then there exists matrices $\bar{N}$ and $\bar{L}$ of compatible dimensions such that the following system (16) is a reduced-order observer for the system (1).*

$$\begin{aligned}\dot{z} &= \bar{N}z + Bu + \bar{L}y\\ \hat{x} &= Rz + My.\end{aligned} \tag{16}$$

where $z \in \mathbb{R}^{n_0}$. Rest of the matrices are same as defined in the previous subsection 3.1.

Proof Assume the existence of matrices $\bar{N}$, $\bar{K}$ and $\bar{L}$ such that

$$\bar{N} = AR - \bar{K}CR \tag{17}$$

$$\bar{L} = AM + \bar{K} - \bar{K}CM \tag{18}$$

From equations (16) and (2) the error vector

$$\begin{aligned}e &= x - \hat{x}\\ &= x - Rz - MCx\\ &= (I - MC)x - Rz\\ &= R(Ex - z).\end{aligned} \tag{19}$$

Let $e_1 = Ex - z$. Then

$$\begin{aligned}\dot{e_1} &= E\dot{x} - \dot{z} \\ &= Ax + Bu - (\bar{N}z + Bu + \bar{L}Cx) \\ &= (A - \bar{L}C)x - \bar{N}(Ex - e_1) \\ &= \bar{N}e_1 + (A - \bar{L}C - \bar{N}E)x. \end{aligned} \tag{20}$$

From the equations (17) and (18), we have $A - \bar{L}C - \bar{N}E = 0$. Thus the equation (20) follows

$$e_1 = \exp(\bar{N})(Ex(0) - z(0)),$$

and

$$e = R\exp(\bar{N})(Ex(0) - z(0)). \tag{21}$$

This dynamics is stable if and only if matrix $\bar{N}$ is stable. Which follows from the detectability of matrix pair (AR, CR) and equation (17). Hence the Theorem is proved.

Remark 2 In the above theorem, the order of the proposed reduced-order observer is n_0, i.e., the dynamical order of the given system (1). It is clear from equation (4) that $n_0 \geq n - rank(C)$. If equality holds, order of the proposed observer will be equal to order of reduced-order observer given in [4].

Now, we again summarize the procedure for designing the reduced-order observer in the following algorithm.

Algorithm 2. For Reduced-order observer for the system (1)

1. Repeat steps 1-3 of full-order observer design Algorithm 1.
2. Find matrix $\bar{K}$ by pole placement or LMI approach such that $\bar{N} = AR - \bar{K}CR$ is a stable matrix.
3. Calculate $\bar{L} = AM + \bar{K} - \bar{K}CM$.

4 Numerical Examples

Example 1 Consider the descriptor system (1) described by the following matrices:

$$\tilde{E} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -1 & 5 & -5 \\ -7 & 7 & -8 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, and \quad \tilde{C} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

First, we reduce this system in the form of system (2) by calculating

$$Q = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix},$$

$$E = \begin{bmatrix} -2.2361 & -4.4721 & -8.9443 \end{bmatrix},$$

$$A = \begin{bmatrix} 6.7082 & -8.4971 & 9.3915 \end{bmatrix}, \quad B = \begin{bmatrix} -2.2361 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.2361 & -1.3416 & 0.8944 \\ 1.0000 & 0 & 1.0000 \end{bmatrix}.$$

Since rank $\begin{bmatrix} E \\ C \end{bmatrix} = 3$, using the algorithm given in the Appendix, we calculate

$$R = \begin{bmatrix} 0.0583 \\ -0.1361 \\ -0.0583 \end{bmatrix}, \quad RE = \begin{bmatrix} -0.1304 & -0.2609 & -0.5217 \\ 0.3043 & 0.6087 & 1.2174 \\ 0.1304 & 0.2609 & 0.5217 \end{bmatrix},$$

and $RA = \begin{bmatrix} 0.3913 & -0.4957 & 0.5478 \\ -0.9130 & 1.1565 & -1.2783 \\ -0.3913 & 0.4957 & -0.5478 \end{bmatrix}$.

Now, we can check that rank $\begin{bmatrix} I - RE \\ C \end{bmatrix} = rank(C) = 2$.

Since rank $\begin{bmatrix} \lambda I - RA \\ C \end{bmatrix} \neq 3$, for $\lambda = 1$, so matrix pair (RA, C) is not detectable. Hence, the given system is also not detectable and value of λ for which condition $(H2)$ is not satisfied is $\lambda = 1$.

Example 2 Consider (1) described by the following matrices (This example is taken from [19])

$$\tilde{E} = \begin{bmatrix} 0.5 & -2.5 & 0 \\ 3 & -3 & 4 \\ 2 & -1 & 3 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -1 & 4.5 & -0.5 \\ -7 & 7 & -8 \\ -5 & 3 & -6 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad u = \begin{bmatrix} e^{-t}sint \\ 0.2sin2t \\ 0.2sin3t \end{bmatrix}$$

and $\tilde{C} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$.

Then as per Algorithm 1,

$$Q = \begin{bmatrix} -0.2163 & -0.8299 & -0.5143 \\ 0.9029 & 0.0304 & -0.4287 \\ 0.3714 & -0.5571 & 0.7428 \end{bmatrix},$$

$$E = \begin{bmatrix} -3.6264 & 3.5447 & -4.8625 \\ -0.3148 & -1.9197 & -1.1646 \end{bmatrix},$$

$$A = \begin{bmatrix} 8.5970 & -8.3254 & 9.8331 \\ 1.0280 & 2.9897 & 1.8778 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.7306 & -0.8299 & -0.5319 \\ 0.4742 & 0.0304 & 1.3620 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.1857 & 0 & -0.1857 \\ 1.0000 & 0 & 1.0000 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.3565 & 0.6582 \\ 0.1578 & -0.2295 \\ -0.3565 & -0.6582 \end{bmatrix},$$

and

$$M = \begin{bmatrix} -0.4488 & 2.4167 \\ -0.0898 & 0.4833 \\ 0.2693 & -1.4500 \end{bmatrix}.$$

Full-order observer design:
Since matrix (RA, C) is detectable, using MATLAB LMI tool box, we calculate
$K = \begin{bmatrix} 50.8321 & 13.9307 \\ 41.3786 & 8.8045 \\ -23.9634 & -8.4413 \end{bmatrix}$.
Then by equations (12)-(13),
$N = \begin{bmatrix} -0.75 & -1 & 0.25 \\ 0 & -2 & 0 \\ 0.25 & 1 & -0.75 \end{bmatrix}$
and $L = \begin{bmatrix} 51.3257 & 11.2723 \\ 41.5581 & 7.8378 \\ -24.3673 & -6.2663 \end{bmatrix}$.
If we take $x(0) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, $z(0) = \begin{bmatrix} 10 & 11 & 6 \end{bmatrix}^T$, simulation results are plotted in Figure 1, which reveals that the estimated values of the states follow the **true** states well.

Reduced-order observer design:
For $x(0) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, $z(0) = \begin{bmatrix} 10 & 11 \end{bmatrix}^T$, Algorithm 2 gives,
$\bar{K} = \begin{bmatrix} 64.7015 & 22.5099 \\ 56.2743 & 23.3283 \end{bmatrix}$, $\bar{N} = \begin{bmatrix} -1.7544 & 1.0973 \\ 0.1688 & -1.2456 \end{bmatrix}$,
$\bar{L} = \begin{bmatrix} 66.1223 & 14.8588 \\ 58.3620 & 12.0857 \end{bmatrix}$.
Simulation results are plotted in Figure 2.

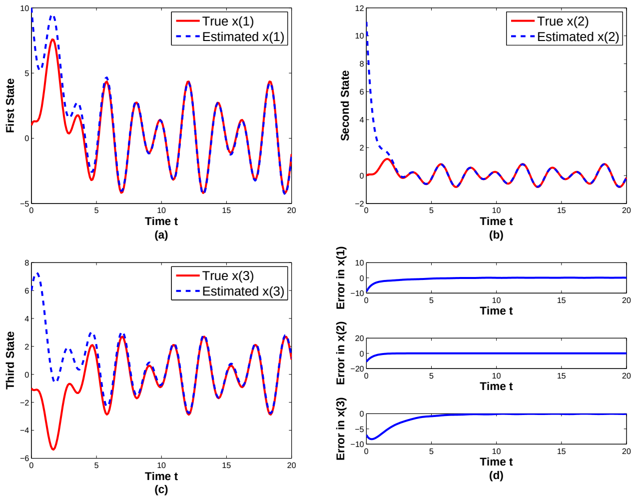

Fig. 1 (a)-(c) Plot of **true** and estimated value of states by full-order observer in Example 2. (d) Plot of errors in states by full-order observer in Example 2.

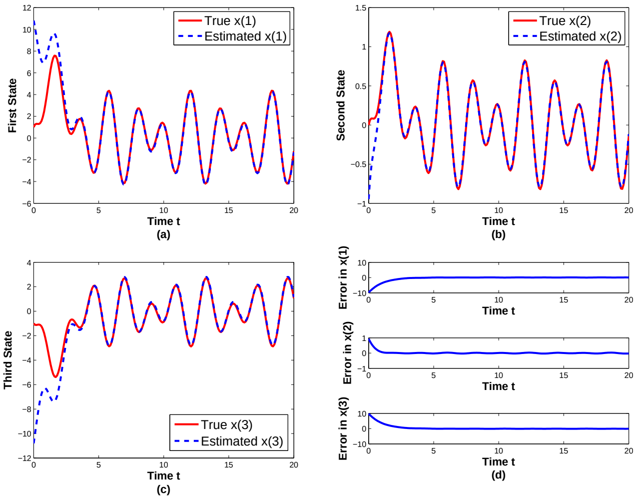

Fig. 2 (a)-(c) Plot of **true** and estimated value of states by reduced-order observer in Example 2. (d) Plot of errors in states by reduced-order observer in Example 2.

Example 3 Consider (1) described by the following matrices (This example is taken from [4])

$$\tilde{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } u = t^2.$$

Then as per Algorithm 1,

$$Q = I_4,$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0.6265 \\ 0 & -0.8440 \\ -1 & 0.9598 \\ 1 & 0.3333 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & -0.6265 \\ -0 & 0 & 1.8440 \\ -1 & 1 & -1.9598 \\ 1 & 0 & -0.3333 \end{bmatrix}.$$

Full-order observer design: Since matrix (RA, C) is observable, by pole placement technique for poles $[-1, -2, -3, -4]$, we calculate

$$K = \begin{bmatrix} -2.7031 & 2.0302 & -2.2363 \\ 0.6138 & -0.7490 & 4.4640 \\ -1.5056 & 1.5606 & -2.0025 \\ 0.2045 & 1.4432 & -1.9992 \end{bmatrix},$$
$$N = \begin{bmatrix} -2.0767 & 0.2061 & -1.0302 & 0.6729 \\ -0.2302 & -3.7150 & 0.7490 & 0.1352 \\ -0.5458 & 0.4419 & -2.5606 & -0.0550 \\ 0.5378 & 0.5560 & -0.4432 & -1.6477 \end{bmatrix}, \text{ and}$$
$$L = \begin{bmatrix} -1 & 1 & 1.2394 \\ 0 & 0 & -3.7553 \\ 1 & -1 & 4.1909 \\ -1 & 1 & 0.1070 \end{bmatrix}.$$

If we take $x(0) = \begin{bmatrix}1 & 0 & -1 & 1\end{bmatrix}^T$, $z(0) = \begin{bmatrix}10 & 11 & 6 & 8\end{bmatrix}^T$, simulation results are plotted in Figure 3, which reveals that the estimated values of the states follow the true states well.

Reduced-order observer design:
As per Algorithm 2, using MATLAB LMI tool box, calculate
$$\bar{K} = \begin{bmatrix} 8.1448 & -27.6509 & -20.0201 \\ 43.5506 & -19.5027 & -26.6451 \end{bmatrix}, \; \bar{N} = \begin{bmatrix} -1 & -1.1324 \\ 1 & -0.3382 \end{bmatrix},$$
$$\bar{L} = \begin{bmatrix} -1 & 1 & 0.1324 \\ 0 & 0 & 0.3382 \end{bmatrix}.$$
For $x(0) = \begin{bmatrix}1 & 0 & -1 & 1\end{bmatrix}^T$, $z(0) = \begin{bmatrix}10 & 11\end{bmatrix}^T$, simulation results are plotted in Figure 4.

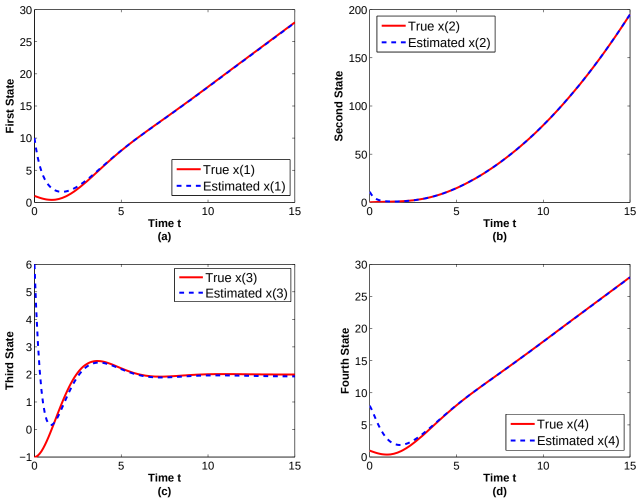

Fig. 3 (a)-(d) Plot of true and estimated value of states by full-order observer in Example 3.

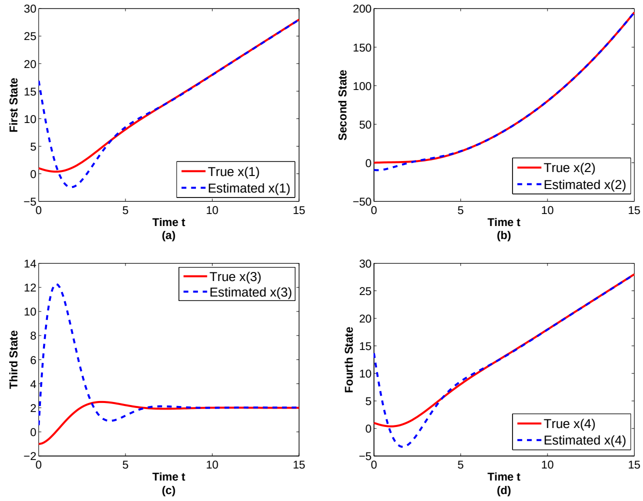

Fig. 4 (a)-(d) Plot of true and estimated value of states by reduced-order observer in Example 3.

Effectiveness of the proposed method can be seen through this example since estimated states converge to the true states even in the presence of unbounded input t^2.

5 Conclusion

We have presented a method to check the detectability of any rectangular system under the assumption of I-observability. Moreover, if the given system is not detectable then proposed approach finds a λ such that the condition (H2) does not hold. Using detectability results, methods have been developed to design full- and reduced-order state observers for the given rectangular descriptor system. To design observers, sufficient condition in terms of the given system operators is equivalent to those in [4] and [13]. But, the technique developed in this paper presents a straightforward method because many of the matrices used in the design of full- and reduced-order observers are same as explained in Algorithms 1 and 2. The results have been illustrated via some numerical examples.

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Appendix A. Algorithm to find matrix $\boldsymbol{R}$:

1. Determine
$p_1 :=$ rank of matrix C
$n_0 \times n$:=order of matrix E.

2. Check rank $\begin{bmatrix} E \\ C \end{bmatrix} = n$, then go to steps 3-8.

3. Carry out the singular value decomposition (SVD) of matrix $C = U_1 \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T$.

4. Calculate $P = V_1 \begin{bmatrix} D_1^{-1} & 0 \\ 0 & I_{n-p_1} \end{bmatrix}$.

5. Calculate $E_2 = EP \begin{bmatrix} 0_{p_1 \times (n-p_1)} \\ I_{n-p_1} \end{bmatrix}$.

6. Carry out the SVD of matrix $E_2 = U_2 \begin{bmatrix} D_2 \\ 0 \end{bmatrix} V_2^T$.

7. Calculate $R_0 = \begin{bmatrix} 0 & I_{n_0+p_1-n} \\ V_2 D_2^{-1} & 0 \end{bmatrix} U_2^T$.

8. Calculate $R = P \begin{bmatrix} 0_{(n-n_0) \times n_0} \\ R_0 \end{bmatrix}$.

References

1. Campbell SL (1980) Singular systems of differential equations, vol 40. Pitman
2. Dai L (1989) The difference between regularity and irregularity in singular systems. Circuits Syst Signal Process 8(4):435–444
3. Dai L (1989) Singular control systems, vol 118. Springer-Verlag
4. Darouach M, Boutayeb M (1995) Design of observers for descriptor systems. IEEE Trans Autom Control 40(7):1323–1327
5. Darouach M, Zasadzinski M, Hayar M (1996) Reduced-order observer design for descriptor systems with unknown inputs. IEEE Trans Autom Control 41(7):1068–1072
6. Duan GR (2010) Analysis and design of descriptor linear systems, vol 10. Springer
7. Feng Je, Lam J, Xu S (2009) Finite-time functional observers for descriptor systems. Int J Control Autom Syst 7(3):341–347
8. Gahinet P, Nemirovskii A, Laub AJ, Chilali M (1994) The LMI control toolbox. In: Decision and Control, IEEE, vol 2, pp 2038–2038
9. Gupta MK, Tomar NK, Bhaumik S (2014) Detectability and observer design for linear descriptor system. In: 22nd Mediterranean Conference on Control and Automation, IEEE, pp 1094–1098
10. Gupta MK, Tomar NK, Bhaumik S (2014) On observability of irregular descriptor systems. In: Advances in Control and Optimization of Dynamical Systems, IFAC, vol 3, pp 376–379
11. Gupta MK, Tomar NK, Bhaumik S (2014) PD observer design for linear descriptor systems. In: International Conference on Mathematical Sciences, Elsevier, pp 40–43
12. Hou M, Müller PC (1995) Design of a class of Luenberger observers for descriptor systems. IEEE Trans Autom Control 40(1):133–136
13. Koenig D, Mammar S (2002) Design of proportional-integral observer for unknown input descriptor systems. IEEE Trans Autom Control 47(12):2057–2062
14. Kumar A, Daoutidis P (1999) Control of nonlinear differential algebraic equation systems with applications to chemical processes, vol 397. CRC Press
15. Lewis F (1990) Geometric design techniques for observers in singular systems. Automatica 26(2):411–415
16. Minamide N, Arii N, Uetake Y (1989) Design of observers for descriptor systems using a descriptor standard form. Int J Control 50(6):2141–2149
17. Müller PC, Hou M (1991) On the observer design for descriptor systems. In: Decision and control, IEEE, pp 1960–1961
18. Müller PC, Hou M (1993) On the observer design for descriptor systems. IEEE Trans Autom Control 38(11):1666–1671
19. Ren J, Zhang Q (2010) PD observer design for descriptor system: An LMI approach. Int J Control Autom Syst 8(4):735–740
20. Riaza R (2008) Differential-algebraic systems: Analytical aspects and circuit applications. World Scientific
21. Shafai B, Carroll R (1986) Design of minimal-order observer for singular systems. In: American Control Conference, IEEE, pp 2077–2082
22. Sinha A (2007) Linear systems. CRC Press
23. Small SJ (2011) Runge-Kutta type methods for differential-algebraic equations in mechanics. University of Iowa
24. Uetake Y (1989) Pole assignment and observer design for continuous descriptor systems. Int J Control 50(1):89–96
25. Verhaegen M, Van Dooren P (1986) A reduced order observer for descriptor systems. Syst Control Lett 8(1):29–37
26. Wu AG, Duan GR (2006) Design of PI observers for continuous-time descriptor linear systems. IEEE Trans Syst Man Cybern Part B Cybern 36(6):1423–1431
27. Wu AG, Duan GR (2007) Design of PD observers in descriptor linear systems. Int J Control Autom

Syst 5(1):93–98