Aerospace Science and Technology 27 (2013) 32–43

Contents lists available at SciVerse ScienceDirect

Aerospace Science and Technology

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Multiple target tracking based on homogeneous symmetric transformation of measurements

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A R T I C L E   I N F O

Article history:
Received 2 September 2011
Received in revised form 28 April 2012
Accepted 11 June 2012
Available online 26 June 2012

Keywords:
Multiple target tracking
Symmetric measurement equation
Kalman filter

A B S T R A C T

Symmetrical measurement equation, generated from homogeneous symmetric functions, has been proposed in this paper for tracking multiple targets. The observability condition, resultant measurement noise and its covariance for any number of particles arising from proposed symmetric transformation of measurement have been derived. The derived expression of resultant noise covariance is verified using Monte Carlo run. As a case study, three particles in motion are considered where positions and velocities of the particles are estimated using extended Kalman filter. From the simulation results it is found that the targets' identity is lost during estimation. The target tracks have been labeled by minimizing the sum of square errors over the permutation of states. The performance of estimator in terms of root mean square error is compared with the two types of symmetric transformation of measurements, namely sum of power and sum of product form, existing in literature. Results are also compared with optimal state estimator which assumes that the correct association between measurements and targets is known. From simulation it is observed that RMSEs of position and velocity are small in homogeneous form compared to those obtained from sum of power and product form.

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1. Introduction

The research interest for simultaneous tracking of multiple objects is increasing rapidly as it finds many applications in surveillance [3], robotics [5], collision avoidance [6], econometrics [7] and signal processing just to name a few. The core problem is to track multiple targets in clutter environment where targets may originate or terminate at any instant of time and association between targets and measurements is unknown. Classical approach is to compute the association probabilities [17,18] between measurements and targets before estimation. The main drawback of such approach is its computational inefficiency as the complexity increases exponentially (or factorial) with the number of targets.

In an alternative approach [9–11,13,14], the computational complexity described above can be circumvented if the number of targets to be tracked at any instant is assumed to be known. The assumption may be justified with the availability of sensor (for example Radar) which is capable of collecting the information which can be used to infer the number of targets within the area of coverage. The key idea is to convert the measurement data with unknown association to a symmetrical measurement equation to estimate the states of the targets. In this way it is possible to estimate targets' state without even considering association between targets and measurements. Sometimes this type of filter is called as symmetrical measurement equation (SME) filter [11].

In this paper, a new type of symmetrical measurement transformation based on homogeneous symmetric function has been introduced to transform the measurement data, obtained from sensor to form symmetric measurement equation. The proposed form will be a new addition in the family of symmetrical measurement equations which consists of two types of measurement equations namely sum of power and sum of product form. The observability condition for the developed symmetric transformation of measurements has been derived in the form of a proposition. The expressions of resultant measurement noise and its covariance have been derived and the later has been verified using Monte Carlo run. The approach has been illustrated through a simple case study where motion of three particles is considered. It has been observed that although the targets' states have been estimated, the estimator fails to label the track of particles. To label the track, all the permutations of states have been considered and the estimated values of states are frozen for that permutation which has least sum of square error. A comparison of estimation accuracy among different types of symmetrical measurements and also

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http://dx.doi.org/10.1016/j.ast.2012.06.004

with the associated filter [10,11] (estimator with known correct association) has been made in terms of root mean square error (RMSE). Simulation results reveal that RMSEs of position and velocity are less in homogeneous form compared to that of obtained from sum of power and sum of product form.

It may not be irrelevant to mention here that there is no optimal or the “best” Gaussian filter available in literature for multiple target tracking with nonlinear measurements. The claim of polynomial filter introduced by Luca et al. [15] as optimal estimation approach [2] in target tracking is misleading. In the polynomial filter, first two moments are computed for certain types of polynomial nonlinearity using full Taylor series expansion thus the non-Gaussian pdf arises due to nonlinear process and measurement equations is approximated as Gaussian. The previous study reveals that [14] the performance of SME approach depends on the combination of symmetrical measurement equation and nonlinear estimator rather than on either individually. In this paper a systematic approach has been taken where the performance of different types of symmetrical measurement equation has been compared with the same nonlinear estimator. Obviously the study of different types of symmetrical equation with several other nonlinear filters remains under the scope of future work.

The paper is organized as follows: Section 2 presents the formulation of target tracking problem of N particles. Section 3 is focused on the development of new symmetric measurement equation generated from homogeneous symmetric functions characterized with its resultant measurement noise and its covariance. A case study of three particles in motion is considered as a simulation problem and results are discussed in Section 4. Concluding remarks are in Section 5.

2. Problem formulation

2.1. Process model

Let us consider N particles maneuvering in a three dimensional space. Being interested in estimating position and velocity of the particles, state vector is considered to be constituted with position and velocity of all the particles along three axes. So for N targets moving in space, state vector can be assumed as $X_k = [x_{1k}\ x_{2k}\ \ldots\ x_{NK}\ v_{1k}\ v_{2k}\ \ldots\ v_{Nk}]^T$, where x_{ik} and v_{ik} represent the positions and velocities of ith target at time kT, with T as sampling time and $i = 1, 2, \ldots, N$. For two particles moving along straight line, the state vector would be $X_k = [x_{1k}\ x_{2k}\ v_{1k}\ v_{2k}]^T$. If three particles move in 1D space the state vector would be $X_k = [x_{1k}\ x_{2k}\ x_{3k}\ v_{1k}\ v_{2k}\ v_{3k}]^T$. If two particles move in 2D space, there are horizontal and vertical components of position and velocity. So the state vector would be $X_k = [x_{x,1k}\ x_{y,1k}\ x_{x,2k}\ x_{y,2k}\ v_{x,1k}\ v_{y,1k}\ v_{x,2k}\ v_{y,2k}]^T$, where $x_{x,ik}$ and $v_{x,ik}$ are the x directional position and velocity of ith particle at any instant k. Similar expression of state vector can be obtained for N particles moving in three dimensional space. Depending upon the nature of maneuver, the positions and velocities of the targets are modeled in state space using the nonlinear difference equations

$$X_{k+1} = \gamma(X_k) + Bw_k \tag{1}$$

where $\gamma(.)$ is a nonlinear function and w_k is zero-mean Gaussian white noise with Q_k covariance.

2.2. Measurement model

Now suppose the sensor which provides noisy measurement of the position is located at the origin of the coordinate system. If we assume the sensor outputs are the individual positions of the targets with correctly known association between targets and measurements, the measurement equation becomes linear and can be written as:

$$Y_k = HX_k + u_k \tag{2}$$

Here $Y_k = [y_{1k}\ y_{2k}\ \ldots\ y_{Nk}]^T$ where y_{ik} is the ith sensor measurement data at time kT; H is measurement matrix and $u_k = [u_{1k}\ u_{2k}\ \ldots\ u_{Nk}]^T$ is the measurement noise. For example, if two particles move in 1D space the measurement data would be $Y_k = [y_{1k}\ y_{2k}]^T$. For three particles in 1D space the measurement vector is $Y_k = [y_{1k}\ y_{2k}\ y_{3k}]^T$. Similarly for two particles moving in a plane, the measurement data would be $Y_k = [y_{x,1k}\ y_{y,1k}\ y_{x,2k}\ y_{y,2k}]^T$. Similar expression can be written for any number of particles moving in 3D of space. As stated earlier, considering only targets' position as measurements, measurement matrix becomes $H = [I_{DN}\ 0_N]$ where D is the dimension of the space where the particles are moving and I_{DN} is the DN dimensional unity matrix. We also assume the measurement noise or sensor noise, u_k, is white Gaussian with zero mean and σ_k^2 covariance ($u_k \sim N(0, \sigma_k^2)$). As in this case both the process and measurement equations are linear, the problem can be solved using Kalman filter (KF) [1]. Since the estimator knows the correct association, this filter may also be called as associated filter [10,11].

Now let us consider the scenario where correct data association, i.e. the correct correspondence between the sensor measurements and respective target's position is not known. To circumvent data association problem, the sensor data are transformed through symmetrical transformation to form symmetrical measurement equation which is used by the filter to estimate position and velocity of the targets. In this respect three types of symmetrical transformation have been considered here. Among them, the two forms, sum of power [10] and sum of product [11] have appeared in literature. The homogeneous symmetry has been proposed in this paper.

2.2.1. Sum of power symmetry

The symmetric measurement equation for sum of power form for N particles can be written as

$$Y_k = \left[\sum_{i=1}^{N} y_{ik} \quad \sum_{i=1}^{N} y_{ik}^2 \quad \cdots \quad \sum_{i=1}^{N} y_{ik}^N\right]^T \tag{3}$$

where $y_{ik} = x_{ik} + u_{ik}$ is the measured position of ith particle at time instant kT in presence of noise u_{ik} which is assumed to be white Gaussian with zero mean and σ_k^2 covariance.

Example. For two targets moving in straight line, the sum of power symmetry measurement equation is $Y_k = [y_{1k} + y_{2k} \;\; y_{1k}^2 + y_{2k}^2]^T$. For two targets moving in 2D space, the measurement vector can be written as $Y_k = [y_{x,1k} + y_{x,2k} \;\; y_{y,1k} + y_{y,2k} \;\; y_{x,1k}^2 + y_{x,2k}^2 \;\; y_{y,1k}^2 + y_{y,2k}^2]^T$. Similar expression can be written for any number of particles moving in space.

2.2.2. Sum of product symmetry

Another kind of symmetrical measurement equation is the sum of product form [11].

$$Y_k = \left[\sum_{i=1}^{N} y_{ik} \quad \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} y_{ik} y_{jk} \quad \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^{N} y_{ik} y_{jk} y_{lk} \quad \cdots \quad \prod_{i=1}^{N} y_{ik} \right]^T \tag{4}$$

Example. For two targets moving in 1D space, the sum of product symmetry measurement vector is $Y_k = [y_{1k} + y_{2k} \;\; y_{1k} y_{2k}]^T$. For two targets moving in a plane, the measurement vector can be written as $Y_k = [y_{x,1k} + y_{x,2k} \;\; y_{y,1k} + y_{y,2k} \;\; y_{x,1k} y_{x,2k} \;\; y_{y,1k} y_{y,2k}]^T$. Similarly the measurement vector can be written for any number of particles moving in 1D, 2D or 3D spaces.

It has also been proved in [10,11] that for both type of symmetric measurement equations, the system is observable.

2.2.3. Homogeneous symmetric form

The proposed symmetric measurement equation using homogeneous symmetric function [16] for N particles can be constructed as

$$\begin{aligned} Y_k = \Bigg[& \sum_{j_1=1}^{N} y_{j_1 k}^N \quad \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N} y_{j_1 k}^{N-1} y_{j_2 k} \quad \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-1} \sum_{\substack{j_3=j_2+1 \\ j_3 \neq j_1}}^{N} y_{j_1 k}^{N-2} y_{j_2 k} y_{j_3 k} \quad \cdots \\ & \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-i+2} \cdots \sum_{\substack{j_i=j_{i-1}+1 \\ j_i \neq j_1}}^{N} y_{j_1 k}^{N-i+1} y_{j_2 k} \ldots y_{j_i k} \quad \cdots \quad \prod_{i=1}^{N} y_{ik} \Bigg]^T \end{aligned} \tag{5}$$

It should be noted that the sum of power and sum of product type of symmetric measurement described in [10,11] also consists of homogeneous functions but its degree varies with the element of the measurement vector. In homogeneous type of symmetric measurement, the degree of all elements of measurement vector is the same and equal to the number of particles considered.

Example. For two targets moving in 1D space, the homogeneous symmetric form of measurement vector is $Y_k = [y_{1k}^2 + y_{2k}^2 \;\; y_{1k} y_{2k}]^T$. For two targets moving in 2D space, the measurement vector can be written as $Y_k = [y_{x,1k}^2 + y_{x,2k}^2 \;\; y_{y,1k}^2 + y_{y,2k}^2 \;\; y_{x,1k} y_{x,2k} \;\; y_{y,1k} y_{y,2k}]^T$. Similarly the measurement data can be obtained for any number of particles moving in 1D, 2D or 3D spaces.

3. Characterization of homogeneous symmetric measurements

It would be easier to calculate the covariance of resultant noise of symmetrical measurements if the measurements described by Eqs. (3), (4) and (5) can be expressed in the form of $Y_{ik} = g_i(x_{1k}, x_{2k}, \ldots, x_{Nk}) + \eta_{ik}$. Proposition 1 provides the expression of η_{ik} for the measurement proposed in (5).

Proposition 1. *The measurement equation described by Eq. (5) can be expressed as* $Y_k = g(x_{1k}, x_{2k}, \ldots, x_{Nk}) + \eta_k$ *where* $g(x_{1k}, x_{2k}, \ldots, x_{Nk})$ *and* $\eta_k = [\eta_{1k} \;\; \eta_{2k} \;\; \cdots \;\; \eta_{ik} \;\; \cdots \;\; \eta_{Nk}]^T$ *are as follows*

$$\begin{aligned} g(x_{1k}, x_{2k} \ldots x_{Nk}) &= \big[g_{1k}(x_{1k} \ldots x_{Nk}) \quad g_{2k}(x_{1k} \ldots x_{Nk}) \quad \cdots \quad g_{ik}(x_{1k} \ldots x_{Nk}) \quad \cdots \quad g_{Nk}(x_{1k} \ldots x_{Nk}) \big]^T \\ &= \Bigg[\sum_{j_1=1}^{N} x_{j_1 k}^N \quad \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N} x_{j_1 k}^{N-1} x_{j_2 k} \quad \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-1} \sum_{\substack{j_3=j_2+1 \\ j_3 \neq j_1}}^{N} x_{j_1 k}^{N-2} x_{j_2 k} x_{j_3 k} \quad \cdots \\ &\qquad \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-i+2} \cdots \sum_{\substack{j_i=j_{i-1}+1 \\ j_i \neq j_1}}^{N} x_{j_1 k}^{N-i+1} x_{j_2 k} \ldots x_{j_i k} \quad \cdots \quad \prod_{i=1}^{N} x_{ik} \Bigg]^T \end{aligned} \tag{6}$$

$$\eta_{ik} = \left[\begin{aligned} & \textstyle\sum_{n=1}^{N-i+1} \frac{1}{n!} \sum_{j_1=1}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^n} u_{j_1 k}^n \\ & + \textstyle\sum_{n=2}^{N-i+2} \frac{1}{n-1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^{n-1} \partial x_{j_2 k}} u_{j_1 k}^{n-1} u_{j_2 k} \\ & + \textstyle\sum_{n=3}^{N-i+3} \frac{1}{n-2!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-1} \sum_{\substack{j_3=j_2+1 \\ j_3 \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^{n-2} \partial x_{j_2 k} \partial x_{j_3 k}} u_{j_1 k}^{n-2} u_{j_2 k} u_{j_3 k} + \cdots \\ & + \textstyle\sum_{n=i}^{N} \frac{1}{n-i+1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-i+2} \cdots \sum_{\substack{j_i=j_{i-1}+1 \\ j_i \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^{n-i+1} \partial x_{j_2 k} \ldots \partial x_{j_i k}} u_{j_1 k}^{n-i+1} u_{j_2 k} \ldots u_{j_i k} \end{aligned} \right] \tag{7}$$

Proof. In mathematical terms, each symmetric measurement Y_{ik}, $i = 1, 2, \ldots, N$ can be expressed as $Y_{ik} = g_i(y_{1k}, y_{2k}, \ldots, y_{Nk})$ or,

$$Y_{ik} = g_i(x_{1k} + u_{1k}, x_{2k} + u_{2k}, \ldots, x_{Nk} + u_{Nk}) \tag{8}$$

Now we would like to express (8) in the form of

$$Y_{ik} = g_i(x_{1k}, x_{2k}, \ldots, x_{Nk}) + \eta_{ik} \tag{9}$$

where η_{ik} is the measurement noise. If we accumulate all the terms without involving the noise u_{ik} under the function $g_i(x_{1k}, x_{1k}, \ldots, x_{Nk})$, it can be expressed by Eq. (6).

In Eq. (8), $g_i(x_{1k} + u_{1k}, x_{2k} + u_{2k}, \ldots, x_{Nk} + u_{Nk})$ can be expanded using Taylor series about vector $x_k = [x_{1k}\ x_{2k}\ \cdots\ x_{Nk}]^T$ as follows:

$$Y_{ik} = g_i(x_{1k}, x_{2k}, \ldots, x_{Nk}) + \sum_{j_1=1}^{N} \frac{\partial g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}} u_{j_1 k} + \frac{1}{2!} \sum_{j_1, j_2=1}^{N} \frac{\partial^2 g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k} \partial x_{j_2 k}} u_{j_1 k} u_{j_2 k} + \cdots$$
$$+ \frac{1}{N!} \sum_{j_1, j_2, \ldots, j_N=1}^{N} \frac{\partial^N g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k} \partial x_{j_2 k} \ldots \partial x_{j_N k}} u_{j_1 k} u_{j_2 k} \ldots u_{j_N k} \tag{10}$$

For the case of homogeneous symmetric form given by (5) the Taylor series expansion (10) reduces to

$$Y_{ik} = g_i(x_{1k}, x_{2k}, \ldots, x_{Nk}) + \sum_{n=1}^{N-i+1} \frac{1}{n!} \sum_{j_1=1}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^n} u_{j_1 k}^n + \sum_{n=2}^{N-i+2} \frac{1}{n-1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^{n-1} \partial x_{j_2 k}} u_{j_1 k}^{n-1} u_{j_2 k}$$
$$+ \sum_{n=3}^{N-i+3} \frac{1}{n-2!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-1} \sum_{\substack{j_3=j_2+1 \\ j_3 \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^{n-2} \partial x_{j_2 k} \partial x_{j_3 k}} u_{j_1 k}^{n-2} u_{j_2 k} u_{j_3 k} + \cdots$$
$$+ \sum_{n=i}^{N} \frac{1}{n-i+1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-i+2} \cdots \sum_{\substack{j_i=j_{i-1}+1 \\ j_i \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \ldots, x_{Nk})}{\partial x_{j_1 k}^{n-i+1} \partial x_{j_2 k} \ldots \partial x_{j_i k}} u_{j_1 k}^{n-i+1} u_{j_2 k} \ldots u_{j_i k} \tag{11}$$

Comparing Eq. (11) with (9) the measurement noise can be expressed as (7). □

Example. For two targets moving in a straight line and the measurement in homogeneous symmetric form the function $g(.)$ can be written as $g(x_{1k} x_{2k}) = [x_{1k}^2 + x_{2k}^2\ \ x_{1k} x_{2k}]^T$. For two particles moving in 1D space, the η_k could be written as $\eta_k = [(u_{1k}^2 + u_{2k}^2 + 2u_{1k} x_{1k} + 2u_{2k} x_{2k})\ (u_{2k} x_{1k} + u_{1k} x_{2k} + u_{1k} u_{2k})]^T$. For three particles moving in a straight line, the expressions of $g(.)$ and η_k are provided in Section 4.

Conjecture 1. *For a function $g(x_{1k}, x_{2k}, \ldots, x_{Nk})$ described by Eq. (6), the determinant of Jacobian matrix*

$$G(x_{1k}, x_{2k}, \ldots, x_{Nk}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_{1k}} & \frac{\partial g_1}{\partial x_{2k}} & \cdots & \frac{\partial g_1}{\partial x_{Nk}} \\ \frac{\partial g_2}{\partial x_{1k}} & \frac{\partial g_2}{\partial x_{2k}} & \cdots & \frac{\partial g_2}{\partial x_{Nk}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_N}{\partial x_{1k}} & \frac{\partial g_N}{\partial x_{2k}} & \cdots & \frac{\partial g_N}{\partial x_{Nk}} \end{bmatrix} \tag{12}$$

can be expressed using Eq. (13), where the function $f(.)$ is a polynomial of degree $N(N-1)/2$ and the exact description of the function depends on the number of targets considered.

$$\left|G(x_{1k}, x_{2k}, \ldots, x_{Nk})\right| = Nf\left(\sum_{j=1}^{N} x_{jk}, \sum_{j_1, j_2=1}^{N} x_{j_1 k} x_{j_2 k}, \ldots, \sum_{j_1, j_2 \ldots j_N=1}^{N} x_{j_1 k} x_{j_2 k} \ldots x_{j_N k}\right) \prod_{1 \leqslant j_1 < j_2 \leqslant N} (x_{j_1 k} - x_{j_2 k}) \tag{13}$$

Validation. The determinant in Eq. (12) can be expressed in terms of *polynomial of* $x_k \times \det(\textit{Vandermonde matrix})$. Here we use the method of induction to validate the conjecture. Also we use the well known theorem that says $\det(Vand(x_1, x_2, \ldots, x_N)) = \prod_{1 \leqslant j_1 < j_2 \leqslant N} (x_{j_1 k} - x_{j_2 k})$ [20].

For two particles, $N = 2$,

$$\left|G(x_{1k}, x_{2k})\right| = 2(x_{1k} + x_{2k})(x_{1k} - x_{2k}).$$

For three particles, $N = 3$,

$$\left|G(x_{1k}, x_{2k}, x_{3k})\right| = 3\left(\sum_{j=1}^{3} x_{jk}\right)^3 \prod_{1 \leqslant j_1 < j_2 \leqslant 3} (x_{j_1 k} - x_{j_2 k}).$$

Similarly for $N = 4$,

$$\left|G(x_{1k},\ldots,x_{4k})\right| = 4\Bigg[\left(\sum_{j=1}^{4} x_{jk}\right)^4 \left(\sum_{j_1=1}^{3}\sum_{j_2=j_1+1}^{4} x_{j_1}x_{j_2}\right) + \left(\sum_{j=1}^{4} x_{jk}^2 \sum_{j=1}^{4} x_{jk}\right)\left(\sum_{j=1}^{4} x_{jk}^3 + 3\sum_{j_1=1}^{2}\sum_{j_2=j_1+1}^{3}\sum_{j_3=j_2+1}^{4} x_{j_1}x_{j_2}x_{j_3}\right)\Bigg] \prod_{1\leqslant j_1<j_2\leqslant 4}(x_{j_1k}-x_{j_2k}).$$

For N particles, $G(x_{1k}, x_{2k}, \ldots, x_{Nk})$ can be written as

$$G(x_{1k}, x_{2k}, \ldots, x_{Nk}) = \begin{bmatrix} Nx_{1k}^{N-1} & Nx_{2k}^{N-1} & \cdots & Nx_{ik}^{N-1} & \cdots & Nx_{Nk}^{N-1} \\ \rho_{1k}\sum_{j=2}^{N} x_{jk} + \sum_{j=2}^{N} x_{jk}^{N-1} & \rho_{2k}\sum_{j=1, j\neq 2}^{N} x_{jk} + \sum_{j=1, j\neq 2}^{N} x_{jk}^{N-1} & \cdots & \rho_{ik}\sum_{j=1, j\neq i}^{N} x_{jk} + \sum_{j=1, j\neq i}^{N} x_{jk}^{N-1} & \cdots & \rho_{Nk}\sum_{j=1}^{N-1} x_{jk} + \sum_{j=1}^{N-1} x_{jk}^{N-1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \prod_{j=2}^{N} x_{jk} & \prod_{j=1, j\neq 2}^{N} x_{jk} & \cdots & \prod_{j=1, j\neq i}^{N} x_{jk} & \cdots & \prod_{j=1}^{N-1} x_{jk} \end{bmatrix}$$

where $\rho_{ik} = (N-1)x_{ik}^{N-2}$, $i = 1, 2, 3, \ldots, N$ or,

$$\left|G(x_{1k}, x_{2k}, \ldots, x_{Nk})\right| = Nf\left(\sum_{j=1}^{N} x_{jk}, \sum_{j_1, j_2=1}^{N} x_{j_1k}x_{j_2k}, \ldots, \sum_{j_1, j_2\ldots j_N=1}^{N} x_{j_1k}x_{j_2k}\ldots x_{j_Nk}\right) \prod_{1\leqslant j_1<j_2\leqslant N}(x_{j_1k}-x_{j_2k}) \qquad \square$$

Proposition 2. *The necessary and sufficient condition for the symmetric functional given by (5), to be observable is that the vector* $x_k = [x_{1k}\ x_{2k}\ \ldots\ x_{Nk}]^T$ *should be distinct with nonzero sum of all elements.*

Proof. From (13) it is clear that the rank of Jacobian $G(x_{1k}, x_{2k}, \ldots, x_{Nk})$ is equal to N, whenever x_{ik} are distinct and the vector $x_k = [x_{1k}\ x_{2k}\ \ldots\ x_{Nk}]^T$ has nonzero sum of elements. □

It should be noted that the above described observability type condition is necessary for stable operation of the tracking filter [10,11].

Proposition 3. *The noise vector as described by Eq. (7), can be expressed in the form of* $\eta_k = \sum_{i=1}^{N} \alpha_{ik}(x_k)q_{ik}(u_k)$ *where* $\alpha_{ik}(x_k)$ *is a matrix of dimension* $N \times ({}^{N}C_i N)$ *and* $q_{ik}(u_k)$ *is a column vector of dimension* ${}^{N}C_i N$.

The detailed proof and the expressions of $\alpha_{ik}(x_k)$ and $q_{ik}(u_k)$ have been provided in Appendix A.

3.1. Mean and covariance of noise sequence

The noise for symmetrical measurement would not follow the Gaussian distribution. However the noise sequence, η_k, is approximated as Gaussian and the first two moments have been calculated. The mean could be calculated easily by evaluating $\bar{\eta}_k = E[\eta_k]$. It is also to be noted that if mean of the noise sequence is nonzero, the effective noise sequence is taken after subtracting the $\bar{\eta}_k$ to make it zero mean.

Assuming $\bar{\eta}_k$ is zero, the covariance can be calculated as

$$R_k = E\left[\sum_{j_1=1}^{N}\sum_{j_2=1}^{N} \alpha_{j_1k}(x_k)q_{j_1k}(u_k)q_{j_2k}^T(u_k)\alpha_{j_2k}^T(x_k)\right]$$

Neglecting the error covariance associated with estimated state and assuming x_{ik} and u_{ik} are independent, the expression of covariance can be approximated as

$$R_k \approx \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} \alpha_{j_1k}(\hat{x}_k)E\left[q_{j_1k}(u_k)q_{j_2k}^T(u_k)\right]\alpha_{j_2k}^T(\hat{x}_k) \tag{14}$$

or

$$R_k \approx \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} \alpha_{j_1k}(\hat{x}_k)f_{j_1j_2}(\sigma^2)\alpha_{j_2k}^T(\hat{x}_k)$$

where $f_{j_1j_2}$ is the function needed to be determined for the number of target considered. The evaluated measurement noise covariance matrix for three particles moving in a straight line is expressed in Appendix B.

4. Case study

4.1. Three particles moving in a straight line

4.1.1. Process model

In this subsection, a case study of three particles moving in a straight line in single dimension has been considered. Similar type of problem has been formulated in earlier literatures [9,10]. For N particles, the evolution of position and velocity with time in state space form can be written as:

$$X_{k+1} = FX_k + Bw_k \tag{15}$$

where $F = \begin{bmatrix} I_N & TI_N \\ 0_N & I_N \end{bmatrix}$, $B = \begin{bmatrix} (T^2/2)I_N & 0_N \\ 0_N & I_N \end{bmatrix}$ and T is the sampling time. Assuming the particles are moving in constant velocity and there is no acceleration, process noise can be considered as zero similar to [10].

4.1.2. Measurement model

Linear measurement: For three particles moving in a straight line, the sensor output data for individual position of target will be given by Eq. (2). The initial state values for truth have been taken as $X_0 = [5\ \ 15\ \ 20\ \ 1.5\ \ {-0.5}\ \ {-1}]^T$. Sensor noise ($u_{ik}$) has been assumed to be white Gaussian with zero mean and covariance $\sigma^2 = \text{diag}[25\ \ 25\ \ 25]$. As the process noise covariance (Q) is zero, target velocities remain constant in their respective initial values during the simulation which has been carried out for 20 seconds with the sampling time 0.01 second.

Sum of power form: Symmetric measurement equation in sum of power form for three particles is expressed as:

$$Y_k = \begin{bmatrix} y_{1k} + y_{2k} + y_{3k} & y_{1k}^2 + y_{2k}^2 + y_{3k}^2 & y_{1k}^3 + y_{2k}^3 + y_{3k}^3 \end{bmatrix}^T$$

Sum of product form: Symmetric measurement equation for three particles in sum of product form as described in Eq. (4) is

$$Y_k = [y_{1k} + y_{2k} + y_{3k} \quad y_{1k}y_{2k} + y_{2k}y_{3k} + y_{3k}y_{1k} \quad y_{1k}y_{2k}y_{3k}]^T$$

Detailed expressions of measurement equation, $g(x_{1k}, x_{2k}, x_{3k})$, noise sequence and covariance associated with it for the above two types of symmetrical measurement are provided in [19].

Homogeneous symmetric form: For three particles, the symmetric measurement with noise reduces to $Y_k = [y_{1k}^3 + y_{2k}^3 + y_{3k}^3\ \ y_{1k}^2(y_{2k} + y_{3k}) + y_{2k}^2(y_{1k} + y_{3k}) + y_{3k}^2(y_{1k} + y_{2k})\ \ y_{1k}y_{2k}y_{3k}]^T$. According to Proposition 1, the measurement equation without noise would be $g(x_{1k}, x_{2k}, x_{3k}) = [x_{1k}^3 + x_{2k}^3 + x_{3k}^3\ \ x_{1k}^2(x_{2k} + x_{3k}) + x_{2k}^2(x_{1k} + x_{3k}) + x_{3k}^2(x_{1k} + x_{2k})\ \ x_{1k}x_{2k}x_{3k}]^T$. For this problem $\{\alpha_{1k}, q_{1k}\}$, $\{\alpha_{2k}, q_{2k}\}$ and $\{\alpha_{3k}, q_{3k}\}$ have been calculated using the formula as described above and noise vector is evaluated as

$$\eta_k = \sum_{i=1}^{3} \alpha_{ik}(x_k) q_{ik}(u_{ik}) = \begin{bmatrix} (u_{1k}^3 + u_{2k}^3 + u_{3k}^3 + 3x_{1k}^2u_{1k} + 3x_{2k}^2u_{2k} + 3x_{3k}^2u_{3k} + 3x_{1k}u_{1k}^2 + 3x_{2k}u_{2k}^2 + 3x_{3k}u_{3k}^2) \\ \begin{pmatrix} x_{1k}^2(u_{2k} + u_{3k}) + x_{2k}^2(u_{1k} + u_{3k}) + x_{3k}^2(u_{1k} + u_{2k}) \\ + (u_{1k}^2 + 2x_{1k}u_{1k})(x_{2k} + x_{3k} + u_{2k} + u_{3k}) \\ + (u_{2k}^2 + 2x_{2k}u_{2k})(x_{1k} + x_{3k} + u_{1k} + u_{3k}) \\ + (u_{3k}^2 + 2x_{3k}u_{3k})(x_{1k} + x_{2k} + u_{1k} + u_{2k}) \end{pmatrix} \\ (u_{1k}x_{2k}x_{3k} + u_{2k}x_{1k}x_{3k} + u_{3k}x_{1k}x_{2k} + x_{1k}u_{2k}u_{3k} + x_{2k}u_{3k}u_{1k} + x_{3k}u_{1k}u_{2k} + u_{1k}u_{2k}u_{3k}) \end{bmatrix}$$

The $\bar{\eta}_k$ is calculated as $\bar{\eta}_k = [3\sigma^2(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1})\ \ 2\sigma^2(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1})\ \ 0]^T$. The noise vector is taken as $\eta_k' = (\eta_k - \bar{\eta}_k)$ to make it zero mean. Now to implement any Gaussian filter, the covariance of the noise sequence (R_k) needs to be calculated. The expressions of $R_k = E[\eta_k' {\eta_k'}^T]$ for three particles can be derived from (14). The expression for noise covariance is also verified using Monte Carlo run.

4.2. Particles moving in a plane

In this subsection, two particles moving in a plane have been considered. The state variables are x and y positions and velocities of the particles. So process equation for two particles moving in a plane becomes eighth order and is similar to Eq. (15).

4.2.1. Measurement model

Sum of power form: Symmetric measurement equation in sum of power form for two particles moving in a plane is expressed as $Y_k = [y_{x,1k} + y_{x,2k}\ \ y_{x,1k}^2 + y_{x,2k}^2\ \ y_{y,1k} + y_{y,2k}\ \ y_{y,1k}^2 + y_{y,2k}^2]$ where $y_{x,1k}$ is the noisy measurement of the x position of the first target.

Sum of product form: Symmetric measurement equation with noise in sum of product form for two particles moving in a plane may be expressed as $Y_k = [y_{x,1k} + y_{x,2k}\ \ y_{x,1k}y_{x,2k}\ \ y_{y,1k} + y_{y,2k}\ \ y_{y,1k}y_{y,2k}]$.

Homogeneous symmetric form: Homogeneous symmetric measurement equation can be written as $Y_k = [y_{x,1k}^2 + y_{x,1k}^2\ \ y_{x,1k}y_{x,2k}\ \ y_{y,1k}^2 + y_{y,2k}^2\ \ y_{y,1k}y_{y,2k}]$.

5. Simulation results

5.1. Particles moving in a straight line

As the formulated problem is nonlinear in nature, nonlinear estimators are to be implemented. Although, here the problem has been solved using extended Kalman filter (EKF), and unscented Kalman filter [13,14] other filters like quadrature based filters [4], central

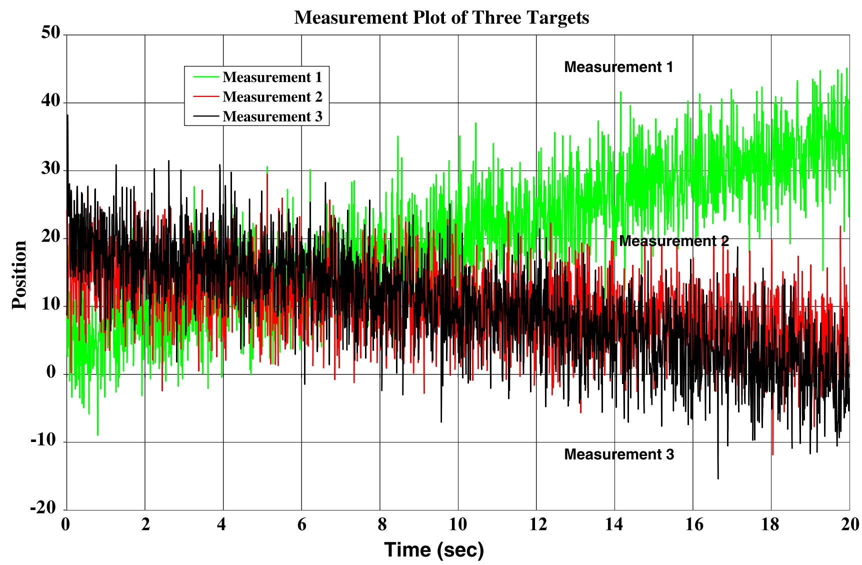

Fig. 1. Measurement plot of three targets' position.

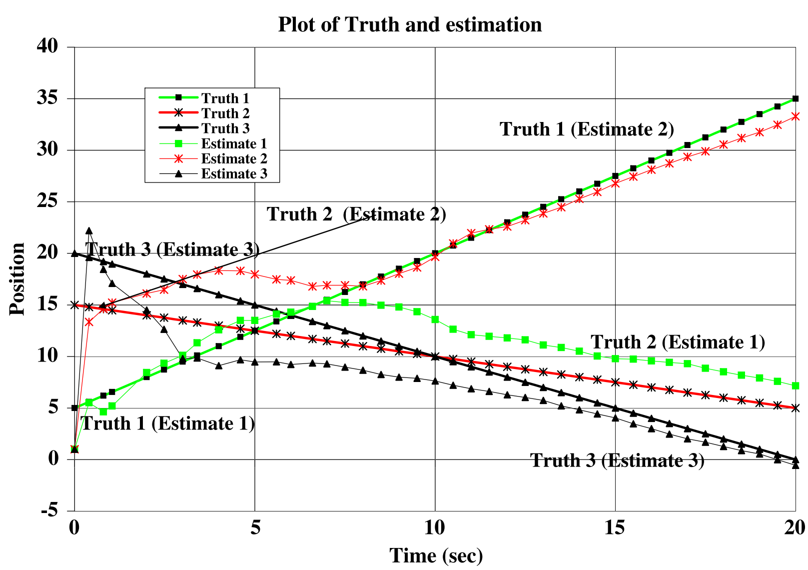

Fig. 2. Truth and estimated values for a single representative run for homogeneous symmetry.

difference filter (CDF) [8], etc. can also be applied without any further calculation of noise parameters. The truth model as well as filter has been simulated in MATLAB environment. The initial state of truth has been taken as $X_0 = [5\ \ 15\ \ 20\ \ 1.5\ \ -0.5\ \ -1]^T$. Sensor noise ($u_{ik}$) has been assumed to be white Gaussian with zero mean and covariance $\sigma^2 = \mathrm{diag}[25\ \ 25\ \ 25]$. The measurement obtained from sensor is plotted in Fig. 1. As the process noise covariance (Q) is zero, target velocities remain constant in their respective initial values during the simulation which has been carried out for 20 seconds with the sampling time 0.01 second. The estimated value of state has been initialized with $\hat{X}_0 = [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]^T$ along with the error covariance $P_0 = \mathrm{diag}[9\ \ 25\ \ 64\ \ 2\ \ 0.25\ \ 16]$. The estimated and truth values of position of three targets for a single representative run have been plotted in Fig. 2 for homogeneous form of symmetrical measurement equation. Similar results are obtained for other two types of symmetrical measurements and have not been shown here. The initial conditions are selected such a way that the targets cross each other within simulation interval. From the simulation, it has been found that the filter tracks the positions of the particles well but without identifying the particles (as shown in Fig. 2), means that tracks are not labeled. Also exchange of track may occur during crossover.

In this paper, a heuristic method has been adopted on the top of SME filter to label the tracks. As we know that the targets are moving in a straight line without any process noise, least square fit of straight line with all permutation of states is performed. The targets have been labeled for the permutation of state for which sum of squared error is minimum. The method can be thought of a kind of smoothing, performed on the top of the SME filtering technique. In Figs. 3 and 4, root mean square error (RMSE) of second target obtained from EKF for 100 Monte Carlo runs has been compared for position and velocity respectively. It can also be observed that in position plot there are kinks near 5th and 10th second which are due to the exchange of track labeling during crossover. From Figs. 3 and 4, it has been observed that the RMSEs of position and velocity are distinctly smaller in sum of product than sum of power form of symmetry, as also explored in [19]; whereas performance of homogeneous symmetric form is comparable to that of sum of product form and sometimes it is better. Higher estimation accuracy in homogeneous symmetry may be due to increased observability happens due

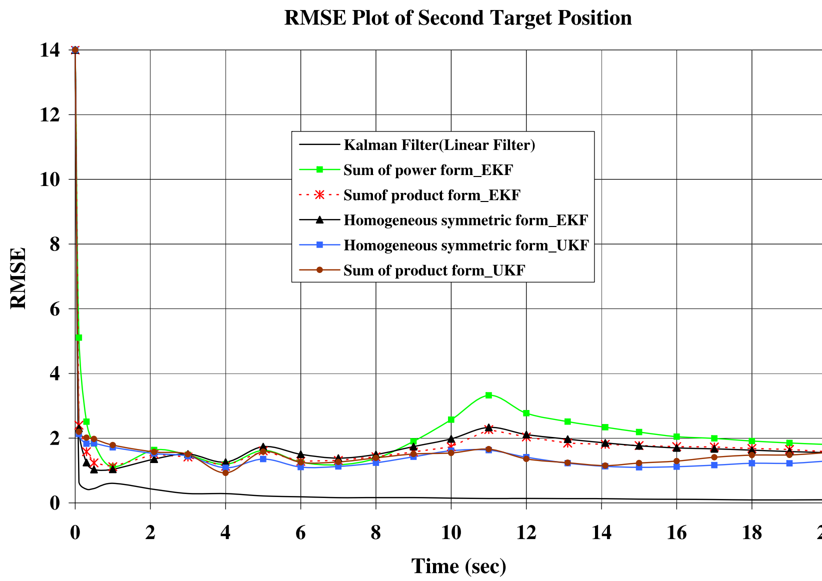

Fig. 3. RMSE plot of position for different symmetrical forms.

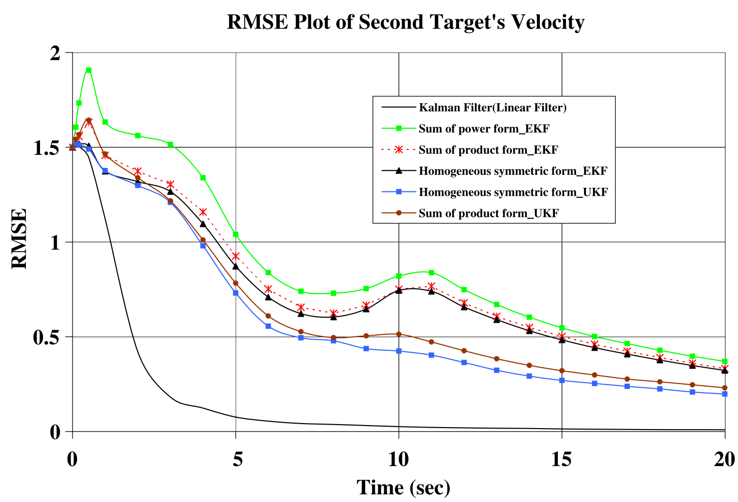

Fig. 4. RMSE plot of velocity for different symmetrical forms.

to incorporation of more interactive terms between the states in measurement equation. Moreover, further investigation reveals that UKF, which approximates the required moments by means of the unscented transform, with homogeneous symmetric form reduces RMSEs of position and velocity significantly and shown in Figs. 3 and 4. Similar types of RMSE plot, obtained for other particles, have not been included here. The performance of different symmetrical forms is also compared with associated filter. As RMSE of associated filter is much lower than EKF, and UKF more advanced nonlinear filters may help for better estimation of targets' position. Also evaluation and comparison of performance with different other forms of symmetric measurement, derived from various symmetric functions (viz. complete homogeneous symmetric functions, Schur functions of different orders) remain under the scope of future work.

5.2. Particles moving in a plane

The problem of target tracking in a plane has been solved using EKF in MATLAB simulation environment. As position and velocity of the x and y coordinates of two particles have been considered, the system becomes eight dimensional in nature. Two particles are assumed to originate at $(5, 4)$ and $(4, 10)$ points in plane and move with uniform velocities $(1.5, 1.5)$ and $(1.3, -0.8)$ respectively. So initial state of truth becomes $X_0 = [5\ 4\ 4\ 10\ 1.5\ 1.5\ 1.3\ -0.8]^T$. Sensors measure x and y coordinate of position with the additive noise u_{ik}, assumed to be white Gaussian with zero mean and covariance $\sigma^2 = \text{diag}[25\ 25\ 25\ 25]$. The estimated value of state has been initialized with $\hat{X}_0 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$ along with the initial error covariance $P_0 = \text{diag}[25\ 16\ 16\ 81\ 0.25\ 2.25\ 0.09\ 1]$. The initial conditions are selected in such a way that the targets cross each other within simulation interval. The estimation of state has been carried out for 15 seconds with the sampling time 0.01 second with three different symmetrical measurement equations described earlier. The truth and estimated trajectories of two targets obtained from homogeneous symmetric measurement have been plotted in Fig. 5. With the described symmetrical form of measurement equations track labeling may be lost. In addition to that, the estimation may pair with one particle's x position and other particle's y position. The phenomenon has been identified earlier and named as "coordinate switching" leading to track "ghost targets". To circumvent the problem, Leven [12] proposed an idea where observed one x coordinate is encoded

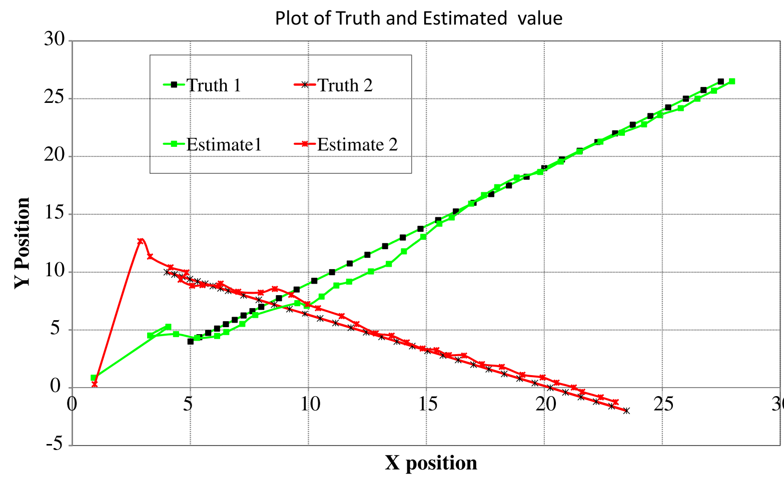

Fig. 5. Truth and estimated values of two particles moving in a plane using homogeneous symmetric form.

in the real part of a complex number and the other coordinate as the imaginary part. To avoid the "coordinate switching" problem, the same type of encoding is possible for homogeneous symmetric transformation of measurement and the study of performance under such encoding scheme needs to be done rigorously in future.

6. Discussions and conclusion

In this work, a new symmetric measurement equation is developed from homogeneous symmetric functions to overcome the data association problem for multiple target tracking. The noise vector and its covariance have been calculated for any number of particles in motion. The observability condition for homogeneous symmetric measurement equation has been derived. Case studies for targets moving in one and two dimensional space have been included. The targets' positions and velocities have been calculated using extended Kalman filter for three types of symmetrical measurement, namely sum of power, sum of product and homogeneous symmetry. The results among the three types of symmetrical measurement along with the associated filter have been compared in terms of RMSE. As RMSE of associated filter is much lower than EKF, and UKF, more advanced nonlinear filter may produce better estimation of targets' position. As the resultant noise covariance has been calculated mathematically, other types of advanced nonlinear Gaussian filter such as Gauss Hermite filter (GHF), central difference filter (CDF), etc., may be implemented easily for better estimation of states. Also other forms of symmetric measurement may be derived from various symmetric functions (viz. complete homogeneous symmetric functions, Schur functions of different orders) for better performance. The proposed homogeneous form will become a new candidate in the family of symmetrical transformation of sensor measurements, used for multiple target tracking to overcome data association problem.

Acknowledgements

The co-author acknowledges the partial financial support provided by the Department of Science and Technology, India for carrying out the research. The authors would like to thank anonymous reviewers for their insightful comments.

Appendix A

Proof of Proposition 3. Using simple algebraic manipulation, the expression (7) can be written as $\eta_k = \sum_{i=1}^{N} \alpha_{ik}(x_k) q_{ik}(u_k)$, where

$$\alpha_{1k}(x_k) = \begin{bmatrix} \frac{\partial g_1}{\partial x_{1k}} & \frac{\partial g_1}{\partial x_{2k}} & \cdots & \frac{\partial g_1}{\partial x_{Nk}} & \frac{\partial^2 g_1}{2!\partial x_{1k}^2} & \frac{\partial^2 g_1}{2!\partial x_{2k}^2} & \cdots & \frac{\partial^2 g_1}{2!\partial x_{Nk}^2} & \cdots & \frac{\partial^N g_1}{N!\partial x_{1k}^N} & \frac{\partial^N g_1}{N!\partial x_{2k}^N} & \cdots & \frac{\partial^N g_1}{N!\partial x_{Nk}^N} \\ \frac{\partial g_2}{\partial x_{1k}} & \frac{\partial g_2}{\partial x_{2k}} & \cdots & \frac{\partial g_2}{\partial x_{Nk}} & \frac{\partial^2 g_2}{2!\partial x_{1k}^2} & \frac{\partial^2 g_2}{2!\partial x_{2k}^2} & \cdots & \frac{\partial^2 g_2}{2!\partial x_{Nk}^2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_N}{\partial x_{1k}} & \frac{\partial g_N}{\partial x_{2k}} & \cdots & \frac{\partial g_N}{\partial x_{Nk}} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $q_{1k}(u_k) = [u_{1k}\ u_{2k}\ \cdots\ u_{Nk}\ u_{1k}^2\ u_{2k}^2\ \cdots\ u_{Nk}^2\ \cdots\ u_{1k}^N\ u_{2k}^N\ \cdots\ u_{Nk}^N]^T$.

$\alpha_{1k}(x_k)$ and $q_{1k}(u_k)$ are matrices of dimension $N \times N^2$ and $N^2 \times 1$ respectively. The 2nd term is

$$\alpha_{2k}(x_k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \frac{\partial^2 g_2}{\partial x_{1k}\partial x_{2k}} & \frac{\partial^2 g_2}{\partial x_{1k}\partial x_{3k}} & \cdots & \frac{\partial^2 g_2}{\partial x_{(N-1)k}\partial x_{Nk}} & \frac{\partial^3 g_2}{2!\partial x_{1k}^2\partial x_{2k}} & \frac{\partial^3 g_2}{2!\partial x_{1k}^2\partial x_{3k}} & \cdots & \frac{\partial^3 g_2}{2!\partial x_{(N-1)k}^2\partial x_{Nk}} & \cdots & \frac{\partial^N g_2}{N-1!\partial x_{1k}^{N-1}\partial x_{2k}} & \frac{\partial^N g_2}{N-1!\partial x_{1k}^{N-1}\partial x_{3k}} & \cdots & \frac{\partial^N g_2}{N-1!\partial x_{(N-1)k}^{N-1}\partial x_{Nk}} \\ \frac{\partial^2 g_3}{\partial x_{1k}\partial x_{2k}} & \frac{\partial^2 g_3}{\partial x_{1k}\partial x_{3k}} & \cdots & \frac{\partial^2 g_3}{\partial x_{(N-1)k}\partial x_{Nk}} & \frac{\partial^3 g_3}{2!\partial x_{1k}^2\partial x_{2k}} & \frac{\partial^3 g_3}{2!\partial x_{1k}^2\partial x_{3k}} & \cdots & \frac{\partial^3 g_3}{2!\partial x_{(N-1)k}^2\partial x_{Nk}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 g_N}{\partial x_{1k}\partial x_{2k}} & \frac{\partial^2 g_N}{\partial x_{1k}\partial x_{3k}} & \cdots & \frac{\partial^2 g_N}{\partial x_{(N-1)k}\partial x_{Nk}} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$q_{2k}(u_k) = \left[u_{1k}u_{2k} \quad u_{1k}u_{3k} \quad \cdots \quad u_{N-1k}u_{Nk} \quad u_{1k}^2 u_{2k} \quad u_{1k}^2 u_{3k} \quad \cdots \quad u_{(N-1)k}^2 u_{Nk} \quad \cdots \quad u_{1k}^{N-1} u_{2k} \quad u_{1k}^{N-1} u_{3k} \quad \cdots \quad u_{(N-1)k}^{N-1} u_{Nk}\right]^T$$

The ith term is

$$\alpha_{ik} = [\theta_i \quad \theta_{i+1} \quad \cdots \quad \theta_N]$$

$$q_{ik}(u_k) = \big[u_{1k}u_{2k}\ldots u_{ik} \quad u_{1k}u_{2k}\ldots u_{(i+1)k} \quad \cdots \quad u_{(N-i+1)k}u_{(N-i+2)k}\ldots u_{Nk} \quad u_{1k}^2u_{2k}\ldots u_{ik} \quad u_{1k}^2u_{2k}\ldots u_{(i+1)k} \quad \cdots$$
$$u_{(N-i+1)k}^2u_{(N-i+2)k}\ldots u_{Nk} \quad \cdots \quad u_{1k}^{N-i+1}u_{2k}\ldots u_{ik} \quad u_{1k}^{N-i+1}u_{2k}\ldots u_{(i+1)k} \quad \cdots \quad u_{(N-i+1)k}^{N-i+1}u_{(N-i+2)k}\ldots u_{Nk}\big]^T$$

where

$$\theta_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^i g_i}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{ik}} & \frac{\partial^i g_i}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{(i+1)k}} & \cdots & \frac{\partial^i g_i}{\partial x_{(N-i+1)k}\partial x_{(N-i+2)k}\ldots\partial x_{Nk}} \\ \frac{\partial^i g_{i+1}}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{ik}} & \frac{\partial^i g_{i+1}}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{(i+1)k}} & \cdots & \frac{\partial^i g_{i+1}}{\partial x_{(N-i+1)k}\partial x_{(N-i+2)k}\ldots\partial x_{Nk}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^i g_N}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{ik}} & \frac{\partial^i g_N}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{(i+1)k}} & \cdots & \frac{\partial^i g_N}{\partial x_{(N-i+1)k}\partial x_{(N-i+2)k}\ldots\partial x_{Nk}} \end{bmatrix}$$

$$\theta_{i+1} = \frac{1}{2!}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^{i+1} g_i}{\partial x_{1k}^2\partial x_{2k}\ldots\partial x_{ik}} & \frac{\partial^{i+1} g_i}{\partial x_{1k}^2\partial x_{2k}\ldots\partial x_{(i+1)k}} & \cdots & \frac{\partial^{i+1} g_i}{\partial x_{(N-i+1)k}^2\partial x_{(N-i+2)k}\ldots\partial x_{Nk}} \\ \frac{\partial^{i+1} g_{i+1}}{\partial x_{1k}^2\partial x_{2k}\ldots\partial x_{ik}} & \frac{\partial^{i+1} g_{i+1}}{\partial x_{1k}^2\partial x_{2k}\ldots\partial x_{(i+1)k}} & \cdots & \frac{\partial^{i+1} g_{i+1}}{\partial x_{(N-i+1)k}^2\partial x_{(N-i+2)k}\ldots\partial x_{Nk}} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\theta_N = \frac{1}{N-i+1!}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^N g_i}{\partial x_{1k}^{N-i+1}\partial x_{2k}\ldots\partial x_{ik}} & \frac{\partial^N g_i}{\partial x_{1k}^{N-i+1}\partial x_{2k}\ldots\partial x_{(i+1)k}} & \cdots & \frac{\partial^N g_i}{\partial x_{(N-i+1)k}^{N-i+1}\partial x_{(N-i+2)k}\ldots\partial x_{Nk}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

It is clear that in $\alpha_{ik}(x_k)$ the elements of the rows up to $(i-1)$ are all zeros and the elements of rest $N-i+1$ rows form the upper triangular matrix. So the Nth term can be written as

$$\alpha_{Nk}(x_k) = \left[0 \quad 0 \quad \cdots \quad \frac{\partial^N g}{\partial x_{1k}\partial x_{2k}\ldots\partial x_{Nk}}\right]^T \quad \text{and} \quad q_{Nk}(u_k) = [u_{1k}u_{2k}\ldots u_{Nk}]^T \qquad \square$$

Appendix B. Calculation of measurement noise covariance

B.1. Sum of power symmetry

In the sum of power symmetry form, the measurement noise covariance (R_k) for three particles in 1D can be calculated assuming x_{ik} and u_{ik} are independent as

$$R_{11k} = 3\sigma^2; \qquad R_{12k} = R_{21k} = 2\sigma^2(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1})$$
$$R_{13k} = R_{31k} = 9\sigma^4 + 3\sigma^2\big(\hat{x}_{1k|k-1}^2 + \hat{x}_{2k|k-1}^2 + \hat{x}_{3k|k-1}^2 + P_{11k} + P_{22k} + P_{33k}\big)$$
$$R_{22k} = 6\sigma^4 + 4\sigma^2\big(\hat{x}_{1k|k-1}^2 + \hat{x}_{2k|k-1}^2 + \hat{x}_{3k|k-1}^2 + P_{11k} + P_{22k} + P_{33k}\big)$$
$$R_{23k} = R_{32k} = 12\sigma^4(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1}) + 6\sigma^2\big(\hat{x}_{1k|k-1}^3 + \hat{x}_{2k|k-1}^3 + \hat{x}_{3k|k-1}^3 + 3(\hat{x}_{1k|k-1}P_{11k} + \hat{x}_{2k|k-1}P_{22k} + \hat{x}_{3k|k-1}P_{33k})\big)$$
$$R_{33k} = 45\sigma^6 + 9\sigma^2\big(\hat{x}_{1k|k-1}^4 + \hat{x}_{2k|k-1}^4 + \hat{x}_{3k|k-1}^4 + 6\big(x_{1k|k-1}^2P_{11k} + x_{2k|k-1}^2P_{22k} + \hat{x}_{3k|k-1}^2P_{33k}\big) + 3\big(P_{11k}^2 + P_{22k}^2 + P_{33k}^2\big)\big)$$
$$+ 36\sigma^4\big(\hat{x}_{1k|k-1}^2 + \hat{x}_{2k|k-1}^2 + \hat{x}_{3k|k-1}^2\big) + 45\sigma^4(P_{11k} + P_{22k} + P_{33k}) + 18\sigma^4(P_{12k} + P_{23k} + P_{13k})$$

where P is the prior error covariance matrix.

B.2. Sum of product symmetry

Similarly for the sum of product kind of symmetry, (R_k) for three particles in 1D can be calculated as

$$R_{11k}, R_{12k} \text{ as above}$$

$$R_{13k} = \sigma^2(\hat{x}_{1k|k-1}\hat{x}_{2k|k-1} + \hat{x}_{2k|k-1}\hat{x}_{3k|k-1} + \hat{x}_{1k|k-1}\hat{x}_{3k|k-1} + P_{12k} + P_{23k} + P_{13k})$$

$$R_{22k} = 3\sigma^4 + 2\sigma^2\big(\hat{x}^2_{1k|k-1} + \hat{x}^2_{2k|k-1} + \hat{x}^2_{3k|k-1} + \hat{x}_{1k|k-1}\hat{x}_{2k|k-1} + \hat{x}_{2k|k-1}\hat{x}_{3k|k-1} + \hat{x}_{1k|k-1}\hat{x}_{3k|k-1} + P_{11k} + P_{22k} + P_{33k} + P_{12k} + P_{23k} + P_{13k}\big)$$

$$R_{23k} = \sigma^2\big((\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1})\big(\hat{x}^2_{3k|k-1} + P_{33k} + 2P_{12k}\big) + (\hat{x}_{2k|k-1} + \hat{x}_{3k|k-1})\big(\hat{x}^2_{1k|k-1} + P_{11k} + 2P_{23k}\big) + (\hat{x}_{3k|k-1} + \hat{x}_{1k|k-1})\big(\hat{x}^2_{2k|k-1} + P_{22k} + 2P_{13k}\big)\big) + \sigma^4(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1})$$

$$R_{33k} = \sigma^2\big[\big\{\big(\hat{x}^2_{1k|k-1} + P_{11k}\big)\big(\hat{x}^2_{2k|k-1} + P_{22k}\big) + 4\hat{x}_{1k|k-1}\hat{x}_{2k|k-1}P_{12k} + 2P^2_{12k}\big\}\big\{\big(\hat{x}^2_{2k|k-1} + P_{22k}\big)\big(\hat{x}^2_{3k|k-1} + P_{33k}\big) + 4\hat{x}_{2k|k-1}\hat{x}_{3k|k-1}P_{23k} + 2P^2_{23k}\big\} + \big\{\big(\hat{x}^2_{1k|k-1} + P_{11k}\big)\big(\hat{x}^2_{3k|k-1} + P_{33k}\big) + 4\hat{x}_{3k|k-1}\hat{x}_{1k|k-1}P_{13k} + 2P^2_{13k}\big\}\big] + \sigma^4\big(\hat{x}^2_{1k|k-1} + \hat{x}^2_{2k|k-1} + \hat{x}^2_{3k|k-1} + P_{11k} + P_{22k} + P_{33k}\big) + \sigma^6$$

B.3. Homogeneous symmetry

For the proposed homogeneous symmetry, the different elements of R_k can be calculated as

$$R_{11k} = 9\sigma^2 E\left[\sum_{i=1}^{3} x^4_{ik}\right] + 45\sigma^4 E\left[\sum_{i=1}^{3} x^2_{ik}\right] + 18\sigma^4 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}x_{jk}\right] - 9\sigma^4\left[\sum_{i=1}^{3}\hat{x}_{ik|k-1}\right]^2 + 45\sigma^6$$

$$R_{12k} = R_{21k} = 6\sigma^2 E\left[\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{3} x^3_{ik}x_{jk}\right] + 6\sigma^2 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x^2_{ik}x^2_{jk}\right] + 36\sigma^4 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}x_{jk}\right] + 18\sigma^4 E\left[\sum_{i=1}^{3} x^2_{ik}\right] - 6\sigma^4\left(\sum_{i=1}^{3}\hat{x}_{ik|k-1}\right)\left[\sum_{i=1}^{3}\hat{x}_{ik|k-1}\right]^2 + 18\sigma^6$$

$$R_{13k} = R_{31k} = 3\sigma^2 E\left[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{2}\sum_{\substack{l=j+1\\ l\neq i}}^{3} x^2_{ik}x_{jk}x_{lk}\right] + 3\sigma^4 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}x_{jk}\right]$$

$$R_{22k} = 2\sigma^2 E\left[\sum_{i=1}^{3} x^4_{ik}\right] + 10\sigma^2 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x^2_{ik}x^2_{jk}\right] + 16\sigma^2 E\left[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{2}\sum_{\substack{l=j+1\\ l\neq i}}^{3} x^2_{ik}x_{jk}x_{lk}\right] + 4\sigma^2 E\left[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3} x^3_{ik}x_{jk}\right] + 36\sigma^4 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}x_{jk}\right] + 24\sigma^4 E\left[\sum_{i=1}^{3} x^2_{ik}\right] - 4\sigma^4\left[\sum_{i=1}^{3}\hat{x}_{ik|k-1}\right]^2 + 24\sigma^6$$

$$R_{23k} = R_{32k} = \sigma^2 E\left[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3} x^3_{ik}x_{jk}\right] + 4\sigma^2 E\left[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{2}\sum_{\substack{l=j+1\\ l\neq i}}^{3} x^2_{ik}x_{jk}x_{lk}\right] + 6\sigma^4 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}x_{jk}\right]$$

$$R_{33k} = \sigma^2 E\left[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x^2_{ik}x^2_{jk}\right] + \sigma^4 E\left[\sum_{i=1}^{3} x^2_{ik}\right] + \sigma^6$$

Exact form of the R_k may be evaluated using the following relationships:

$$E\big[x^2_{ik}\big] = \hat{x}^2_{ik|k-1} + P_{iik}; \qquad E\big[x^4_{ik}\big] = \hat{x}^4_{ik|k-1} + 6\hat{x}^2_{ik|k-1}P_{iik} + 3P^2_{iik}$$

$$E[x_{ik}x_{jk}] = \hat{x}_{ik|k-1}\hat{x}_{jk|k-1} + P_{ijk}; \qquad E\big[x^2_{ik}x^2_{jk}\big] = \big(\hat{x}^2_{ik|k-1} + P_{iik}\big)\big(\hat{x}^2_{jk|k-1} + P_{jjk}\big) + 2P^2_{ijk} + 4\hat{x}_{ik|k-1}\hat{x}_{jk|k-1}P_{ijk}$$

$$E\big[x^3_{ik}x_{jk}\big] = \big(\hat{x}^3_{ik|k-1} + 3\hat{x}_{ik|k-1}P_{iik}\big)\hat{x}_{jk|k-1} + 3P_{ijk}\big(\hat{x}^2_{ik|k-1} + P_{iik}\big)$$

$$E\big[x^2_{ik}x_{jk}x_{lk}\big] = \big(\hat{x}^2_{ik|k-1} + P_{iik}\big)(\hat{x}_{jk|k-1}\hat{x}_{lk|k-1} + P_{jlk}) + 2\hat{x}_{ik|k-1}(\hat{x}_{jk|k-1}P_{ilk} + \hat{x}_{lk|k-1}P_{ijk}) + P_{ijk}P_{ilk}$$

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