Multiple Target Tracking Using Homogeneous Symmetric Measurement

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***Abstract*: A new form of symmetrical measurement equation based on homogeneous symmetric function has been proposed for tracking of multiple targets. The observability condition and noise statistics have been derived for proposed measurement equation. A case study of three particles in motion is considered where positions and velocities of the particles are estimated using extended Kalman filter. It has been found that the track labeling is lost during estimation. The target tracks have been labeled by minimizing sum of square errors over the permutation of states. The performance is compared in terms of root mean square error (RMSE) with sum of product and sum of power form of symmetric measurement equations. From simulation it is observed that RMSE of position and velocity in homogeneous symmetric measurement equation are smaller than that of obtained from sum of power and product form.**

Keywords: Multiple target tracking, Symmetric measurement equation, Kalman filter

I. INTRODUCTION

The problem of simultaneous tracking of multiple numbers of objects attracts many researchers due to its various applications in surveillance [1], robotics [2], collision avoidance [3], econometrics [4] and signal processing, *etc*. The core problem is to track multiple targets in clutter environment where targets may originate or terminate at any instant of time. Moreover unknown association between targets and measurements makes it more challenging to the practitioner. Classical approach to solve the problem is to compute the association probabilities [5, 6] between measurements and targets before estimation of targets' state. The main drawback of such approach is its computational inefficiency as the complexity increases exponentially (or factorial) with the number of targets.

To circumvent the above problem, an alternative approach [7-11] is used where association between targets and measurements need not to be known. The key idea behind this approach is to transform the measurement data, obtained from sensor to form symmetric measurement equation. This type of filter is named as symmetrical measurement equation (SME) filter.

In this paper, a new form of symmetric measurement equation based on the homogeneous symmetric function is proposed for tracking of multiple targets. The proposed form will be a new member in the family of symmetrical measurement equations applied for multiple target tracking problems. The observability condition for homogeneous symmetry has been derived in the form of a proposition. It has been argued that the Gaussian noise remains Gaussian even after transformation through symmetric functions. The expressions of mean and covariance of the noise due to transformation have been derived and verified by Monte Carlo run.

A case study of three particles moving in a straight line is considered. The position and velocity of the particles has been estimated using extended Kalman filter. It has been observed that although the targets' states have been estimated with acceptable accuracy, the estimator fails to label the track of particles. To label the track, all the permutations of states have been considered and the estimated values of states are frozen for least sum of square error. A comparison of estimation accuracy among different forms of symmetrical measurement equation has been made in terms of root means square error (RMSE). From simulation it is observed that RMSE of position and velocity in homogeneous form are smaller than that of obtained from sum of power and product form.

The paper is organized as follows: Section 2 presents the formulation of target tracking problem of N number of particles. Next section is focused on the development and characterization of new form of symmetric measurement equation generated from homogeneous symmetric function. A case study of three particles in motion is considered and simulation results are discussed in section 4. Concluding remarks are in section 5.

II. PROBLEM FORMULATION

Let us consider N particles are maneuvering in space. Being interested in measuring position and velocity, we consider the state vector constituted with position and velocity of all the particles. So for N number of targets moving in space, state vector can be assumed as $X_k = [x_{1k} \quad x_{2k} \quad \ldots \quad x_{NK} \quad v_{1k} \quad v_{2k} \quad \ldots \quad v_{Nk}]^T$, where x_{ik} and v_{ik} represent the positions and velocities of i^{th} target at time kT with T as sampling time and i=1, 2… N. The

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evolution of positions and velocities with time in state space may be modeled using the nonlinear difference equations

$$X_{k+1} = \gamma(X_k) + Bw_k \tag{1}$$

Where $\gamma(.)$ is a nonlinear function and w_k is zero mean Gaussian white noise with Q_k covariance.

Symmetrical Measurement:

Now let us assume a sensor is located at the origin of the coordinate system and provides noisy measurement of the position of particles. Also we assume that data association problem is present, *i.e.* the correct correspondence between the sensor measurements and their respective targets' position is not known. As stated earlier, the sensor data are transformed through symmetrical transformation to form symmetrical measurement equation which is to be used to estimate position and velocity of the targets. In this respect three types of symmetrical transformation have been considered here. Among them the two forms namely sum of power [8] and sum of product [9] have appeared in earlier literature. The homogeneous symmetric form has been proposed in this paper.

1) Sum of Power form: The symmetric measurement equation for sum of power form for N number of particles can be written as [8]

$$Y_K = [\sum_{i=1}^{N} y_{ik} \quad \sum_{i=1}^{N} y^2{}_{ik} \quad \dots \quad \sum_{i=1}^{N} y^N{}_{ik}]^T \tag{2}$$

Where $y_{ik} = x_{ik} + u_{ik}$, is the measured position of i^th particle at time instant kT in presence of noise u_{ik} which is assumed to be white Gaussian with zero mean and σ_k^2 covariance. It is clear that the measurement Y_k is a column vector which could be written as $Y_k = [Y_{1k} \quad Y_{2k} \quad . \quad . \quad . \quad Y_{Nk}]^T$

2) Sum of product form: Another form of symmetric measurement equation is sum of product type which is also appeared in [9].

$$Y_k = [\sum_{i=1}^{N} y_{ik} \quad \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} y_{ik}y_{jk} \quad \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N} y_{ik}y_{jk}y_{lk} \quad \dots \quad \prod_{i=1}^{N} y_{ik}]^T \tag{3}$$

It has been proved that for both form of symmetrical measurement equations the system is observable [8, 9].

3) Homogeneous symmetric form: The above two forms of symmetric measurement equations have appeared in earlier literature [7-11]; whereas the homogeneous symmetric form is proposed in this paper. The symmetric measurement equation using homogeneous symmetric function [12] for N number of particles can be constructed as follows:

$$\begin{aligned} Y_k = [&\sum_{j_1=1}^{N} y_{j_1k}^N \quad \sum_{j_1=1}^{N}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N} y_{j_1k}^{N-1} y_{j_2k} \quad \sum_{j_1=1}^{N}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N-1}\sum_{\substack{j_3=j_2+1\\ j_3\neq j_1}}^{N} y_{j_1k}^{N-2} y_{j_2k} y_{j_3k} \quad . \quad . \\ &. \sum_{j_1=1}^{N}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N-i+2} \dots \sum_{\substack{j_i=j_{i-1}+1\\ j_i\neq j_1}}^{N} y_{j_1k}^{N-i+1} y_{j_2k} \dots y_{j_ik} \quad . \quad . \quad . \quad \prod_{i=1}^{N} y_{ik}]^T \end{aligned} \tag{4}$$

It should be noted that in the proposed form, degree of each element of symmetric measurement equation vector is same and forms a homogeneous symmetric function of order N.

III. CHARACTERISATION OF HOMOGENEOUS SYMMETRIC FUNCTION

It would be easier to calculate the covariance of resultant noise of symmetric measurement described by the equations (2), (3) and (4) if these measurements can be expressed in the form of $Y_{ik} = g_i(x_{1k}, x_{2k}, \dots, x_{Nk}) + \eta_{ik}$. The proposition 1 would provide the expression for η_{ik} for the measurement proposed in equation (4).

PROPOSITION 1: The measurement equation described by equation (4) can be expressed as $Y_k = g(x_{1k}, x_{2k}, \dots, x_{Nk}) + \eta_k$ *where* $g(x_{1k}, x_{2k}, \dots, x_{Nk})$ *and* $\eta_k = [\eta_{1k} \quad \eta_{2k} \quad . \quad . \quad . \quad \eta_{ik} \quad . \quad . \quad . \quad \eta_{Nk}]^T$ *are as follows:*

$$\begin{aligned} &g(x_{1k}, x_{2k} \dots x_{Nk}) = \\ &[g_{1k}(x_{1k} \dots x_{Nk}) \quad g_{2k}(x_{1k} \dots x_{Nk}) \quad . \quad . \quad . \quad g_{ik}(x_{1k} \dots x_{Nk}) \quad . \quad . \quad . \quad g_{Nk}(x_{1k} \dots x_{Nk})]^T \\ &= [\sum_{j_1=1}^{N} x_{j_1k}^N \quad \sum_{j_1=1}^{N}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N} x_{j_1k}^{N-1} x_{j_2k} \quad \sum_{j_1=1}^{N}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N-1}\sum_{\substack{j_3=j_2+1\\ j_3\neq j_1}}^{N} x_{j_1k}^{N-2} x_{j_2k} x_{j_3k} \quad . \quad . \\ &. \sum_{j_1=1}^{N}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N-i+2} \dots \sum_{\substack{j_i=j_{i-1}+1\\ j_i\neq j_1}}^{N} x_{j_1k}^{N-i+1} x_{j_2k} \dots x_{j_ik} \quad . \quad . \quad . \quad \prod_{i=1}^{N} x_{ik}]^T \end{aligned} \tag{5}$$

$$\eta_{ik} = \begin{bmatrix} \sum_{n=1}^{N-i+1} \frac{1}{n!} \sum_{j_1=1}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \dots, x_{Nk})}{\partial x_{j_1k}^n} u_{j_1k}^n + \\ \sum_{n=2}^{N-i+2} \frac{1}{n-1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \dots, x_{Nk})}{\partial x_{j_1k}^{n-1} \partial x_{j_2k}} u_{j_1k}^{n-1} u_{j_2k} + \\ \sum_{n=3}^{N-i+3} \frac{1}{n-2!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N-1} \sum_{\substack{j_3=j_2+1\\ j_3\neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \dots, x_{Nk})}{\partial x_{j_1k}^{n-2} \partial x_{j_2k} \partial x_{j_3k}} u_{j_1k}^{n-2} u_{j_2k} u_{j3k} + \dots \\ + \sum_{n=i}^{N} \frac{1}{n-i+1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{N-i+2} \dots \sum_{\substack{j_i=j_{i-1}+1\\ j_i\neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, \dots, x_{Nk})}{\partial x_{j_1k}^{n-i+1} \partial x_{j_2k} \dots \partial x_{j_ik}} u_{j_1k}^{n-i+1} u_{j_2k} \dots u_{j_ik} \end{bmatrix} \tag{6}$$

PROOF: In mathematical terms each symmetric measurement $Y_{ik}, i = 1,2,\dots N$ can be expressed as $Y_{ik} = g_i(y_{1k}, y_{2k}, \dots, y_{Nk})$ or,

$$Y_{ik} = g_i(x_{1k} + u_{1k}, x_{2k} + u_{2k}, \dots, x_{Nk} + u_{Nk}) \tag{7}$$

Now we would like to express (7) in the form of

$$Y_{ik} = g_i(x_{1k}, x_{2k}, ..., x_{Nk}) + \eta_{ik} \tag{8}$$

Where η_k is the measurement noise. If we accumulate all the term without involving the noise u_k under the function $g(x_{1k}, x_{1k} ... x_{Nk})$, it can be expressed by the equation (5).

In equation (7), $g_i(x_{1k}+u_{1k}, x_{2k}+u_{2k}, ..., x_{Nk}+u_{Nk})$ can be expanded into a Taylor series about the vector $x_k = [x_{1k} \quad x_{2k} \quad ... \quad x_{ik}]^T$ as follows:

$$\begin{aligned} Y_{ik} &= g_i(x_{1k}, x_{2k}, ..., x_{Nk}) + \sum_{j_1=1}^{N} \frac{\partial g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k}} u_{j_1 k} \\ &+ \frac{1}{2!} \sum_{j_1, j_2=1}^{N} \frac{\partial^2 g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k} \partial x_{j_2 k}} u_{j_1 k} u_{j_2 k} + ... + \\ &\frac{1}{N!} \sum_{j_1, j_2 ... j_N=1}^{N} \frac{\partial^N g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k} \partial x_{j_2 k} ... \partial x_{j_N k}} u_{j_1 k} u_{j_2 k} ... u_{j_N k} \end{aligned} \tag{9}$$

For the case of homogeneous symmetric form given by (4), the Taylor series expansion reduces to

$$\begin{aligned} Y_{ik} &= g_i(x_{1k}, x_{2k}, ..., x_{Nk}) + \sum_{n=1}^{N-i+1} \frac{1}{n!} \sum_{j_1=1}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k}^n} u_{j_1 k}^n + \\ &\sum_{n=2}^{N-i+2} \frac{1}{n-1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k}^{n-1} \partial x_{j_2 k}} u_{j_1 k}^{n-1} u_{j_2 k} + \\ &\sum_{n=3}^{N-i+3} \frac{1}{n-2!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-1} \sum_{\substack{j_3=j_2+1 \\ j_3 \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k}^{n-2} \partial x_{j_2 k} \partial x_{j_3 k}} u_{j_1 k}^{n-2} u_{j_2 k} u_{j3k} + ... \\ &+ \sum_{n=i}^{N} \frac{1}{n-i+1!} \sum_{j_1=1}^{N} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{N-i+2} ... \sum_{\substack{j_i=j_{i-1}+1 \\ j_i \neq j_1}}^{N} \frac{\partial^n g_i(x_{1k}, x_{2k}, ..., x_{Nk})}{\partial x_{j_1 k}^{n-i+1} \partial x_{j_2 k} ... \partial x_{j_i k}} u_{j_1 k}^{n-i+1} u_{j_2 k} ... u_{j_i k} \end{aligned} \tag{10}$$

Comparing equation (10) with (8) the measurement noise can be expressed as (6). ■

PROPOSITION 2: For a function $g(x_{1k}, x_{2k}, ..., x_{Nk})$ described by equation (5), the determinant of Jacobian matrix,

$$G(x_{1k}, x_{2k}, ..., x_{Nk}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_{1k}} & \frac{\partial g_1}{\partial x_{2k}} & \cdots & \frac{\partial g_1}{\partial x_{Nk}} \\ \frac{\partial g_2}{\partial x_{1k}} & \frac{\partial g_2}{\partial x_{2k}} & \cdots & \frac{\partial g_2}{\partial x_{Nk}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_N}{\partial x_{1k}} & \frac{\partial g_N}{\partial x_{2k}} & \cdots & \frac{\partial g_N}{\partial x_{Nk}} \end{bmatrix} \tag{11}$$

can be expressed by equation (12), wherethe function f(.) is a polynomial of degree N (N-1) / 2 and the exact description of the function depends on the number of targets.

$$\begin{aligned} | G(x_{1k}, x_{2k}, x_{3k}) | &= Nf \, (\sum_{j=1}^{N} x_{jk}, \sum_{j_1, j_2=1}^{N} x_{j_1 k} x_{j_2 k}, ..., \\ &\sum_{j_1, j_2 ... j_N=1}^{N} x_{j_1 k} x_{j_2 k} ... x_{j_N k}) \prod_{1 \le j_1 < j_2 \le N} (x_{j_1 k} - x_{j_2 k}) \end{aligned} \tag{12}$$

PROOF: The determinant in equation (11) can be expressed in terms of polynomial of $x_k \times$ *det (Vandermonde matrix).* Here we prove the proposition by using the method of induction. Determinant of Vandermonde matrix can be expressed as [13].

$$\det(Vand(x_1, x_2, ... x_N)) = \prod_{1 \le j_1 < j_2 \le N} (x_{j_1 k} - x_{j_2 k})$$

For two particle case, N=2, $| G(x_{1k}, x_{2k}) | = 2(x_{1k} + x_{2k})(x_{1k} - x_{2k})$

For three particle case, N=3,

$$| G(x_{1k}, x_{2k}, x_{3k}) | = 3(\sum_{j=1}^{3} x_{jk})^3 \prod_{1 \le j_1 < j_2 \le 3} (x_{j_1 k} - x_{j_2 k})$$

Similarly for N=4

$$\begin{aligned} | G(x_{1k}, ... x_{4k}) | = 4 [&(\sum_{j=1}^{4} x_{jk})^4 (\sum_{j_1=1}^{3} \sum_{j_2=j_1+1}^{4} x_{j_1} x_{j_2}) + (\sum_{j=1}^{4} x_{jk}^2 \sum_{j=1}^{4} x_{jk}) \times \\ &(\sum_{j=1}^{4} x_{jk}^3 + 3 \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \sum_{j_3=j_2+1}^{4} x_{j_1} x_{j_2} x_{j_3})] \prod_{1 \le j_1 < j_2 \le 4} (x_{j_1 k} - x_{j_2 k}) \end{aligned}$$

For N number of particles $G(x_{1k}, x_{2k}, ..., x_{Nk})$ can be written by (13) where $\rho_{ik} = (N-1) x_{ik}^{N-2}$, i=1,2,3,... N and determinant of that by (12). ■

$$G(x_{1k}, x_{2k}, ..., x_{Nk}) = \begin{bmatrix} N x_{1k}^{N-1} & N x_{2k}^{N-1} & \cdots & N x_{ik}^{N-1} & \cdots & N x_{Nk}^{N-1} \\ \rho_{1k} \sum_{j=2}^{N} x_{jk} + \sum_{j=2}^{N} x_{jk}^{N-1} & \rho_{2k} \sum_{\substack{j=1 \\ j \neq 2}}^{N} x_{jk} + \sum_{\substack{j=1 \\ j \neq 2}}^{N} x_{jk}^{N-1} & \cdots & \rho_{ik} \sum_{\substack{j=1 \\ j \neq i}}^{N} x_{jk} + \sum_{\substack{j=1 \\ j \neq i}}^{N} x_{jk}^{N-1} & \cdots & \rho_{Nk} \sum_{j=1}^{N-1} x_{jk} + \sum_{j=1}^{N-1} x_{jk}^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \prod_{j=2}^{N} x_{jk} & \prod_{\substack{j=1 \\ j \neq 2}}^{N} x_{jk} & & \prod_{\substack{j=1 \\ j \neq i}}^{N} x_{jk} & \cdots & \prod_{j=1}^{N-1} x_{jk} \end{bmatrix} \tag{13}$$

PROPOSITION 3: The necessary and sufficient condition for the symmetric functional given by (4) to be observable is that the vector should be distinct with non zero sum of all elements.

PROOF: From (12), it is clear that the rank of Jacobian $G(x_{1k}, x_{2k}, ..., x_{Nk})$ is equal to N whenever x_{ik} are distinct and the vector $x_k = [x_{1k} \quad x_{2k} \quad . \quad . \quad . \quad x_{ik}]^T$ has non zero sum of elements ■

By algebraic manipulation, the noise vector described by equation (6) can be expressed in the form of $\eta_k = \sum_{i=1}^{N} \alpha_{ik}(x_k) q_{ik}(u_k)$ where $\alpha_{ik}(x_k)$ is a matrix of dimension $N \times ({}^N C_i N)$ and $q_{ik}(u_k)$ is a column vector of dimension ${}^N C_i N$.

Mean and Covariances of noise sequence: From central limit theorem [14], it is clear that, noise for symmetrical measurement remains Gaussian after transformation. To implement Gaussian filter, it is necessary to calculate the mean and covariance of the noise sequence η_k. The mean could be calculated easily by evaluating $\overline{\eta}_k = E[\eta_k]$. The covariance of η_k is given as $R_k = E[\eta_k \eta_k^T]$. It is also to be noted that if the mean of the noise sequence ($\overline{\eta}_k$) becomes non zero, $\overline{\eta}_k$ has to be subtracted from noise sequence (η_k) to make it zero mean. Assuming that $\overline{\eta}_k$ is zero, the covariance can be calculated as

$$R_k = E[\sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \alpha_{j_1 k}(x_k) q_{j_1 k}(u_k) q_{j_2 k}^T(u_k) \alpha_{j_2 k}^T(x_k)]$$

Neglecting the error covariance associated with estimated state and assuming x_{ik} and u_{ik} as independent, the expression of covariance can be approximated as

$$R_k \approx \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \alpha_{j_1 k}(\hat{x}_k) E[q_{j_1 k}(u_k) q_{j_2 k}^T(u_k)] \alpha_{j_2 k}^T(\hat{x}_k)$$

or
$$R_k \approx \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \alpha_{j_1 k}(\hat{x}_k) f_{j_1 j_2}(\sigma^2) \alpha_{j_2 k}^T(\hat{x}_k) \qquad (14)$$

where $f_{j_1 j_2}$ is the function need to be determined for the number of targets to be tracked.

IV. Case Study: Three Particles in Motion

In this section three particles moving in a straight line have been considered. For N number of particles the position and velocity with time in state space can be written as

$$X_{k+1} = F X_k + B w_k \qquad (15)$$

where $F = \begin{bmatrix} I_N & T I_N \\ 0_N & I_N \end{bmatrix}$, $B = \begin{bmatrix} (T^2/2) I_N & 0_N \\ 0_N & I_N \end{bmatrix}$, T is sampling time. Assuming that the particles are moving with constant velocity and without acceleration, process noise could be considered as zero.

A. Measurement Model

Sum of power symmetry: Symmetric measurement equation with noise in sum of power symmetry for three particles is expressed as:

$$Y_k = [y_{1k} + y_{2k} + y_{3k} \quad y_{1k}^2 + y_{2k}^2 + y_{3k}^2 \quad y_{1k}^3 + y_{2k}^3 + y_{3k}^3]^T$$

Sum of product symmetry: Symmetric measurement equation with noise for three particles in sum of product form as described in equation (3) is given as

$$Y_k = [y_{1k} + y_{2k} + y_{3k} \quad y_{1k} y_{2k} + y_{2k} y_{3k} + y_{3k} y_{1k} \quad y_{1k} y_{2k} y_{3k}]^T$$

For the detail expression of $g(x_{1k}, x_{2k}, x_{3k})$, noise sequence and covariance for the above two form of symmetric measurement equation, [15] is referred to readers.

Homogeneous symmetry: For three particles, the symmetric measurement equation with noise reduces to

$$Y_k = [y_{1k}^3 + y_{2k}^3 + y_{3k}^3 \quad y_{1k}^2(y_{2k} + y_{3k}) + y_{2k}^2(y_{1k} + y_{3k}) + y_{3k}^2(y_{1k} + y_{2k}) \quad y_{1k} y_{2k} y_{3k}]^T$$

According to proposition 1, the measurement without noise reduces to $g(x_{1k}, x_{2k}, x_{3k})$

$$= [x_{1k}^3 + x_{2k}^3 + x_{3k}^3 \quad x_{1k}^2(x_{2k} + x_{3k}) + x_{2k}^2(x_{1k} + x_{3k}) + x_{3k}^2(x_{1k} + x_{2k}) \quad x_{1k} x_{2k} x_{3k}]^T$$

For this problem {α_{1k}, q_{1k}}, {α_{2k}, q_{2k}} and {α_{3k}, q_{3k}} have been calculated and the noise vector can be expressed

$$\eta_k = \sum_{i=1}^{3} \alpha_{ik}(x_k) q_{ik}(u_{ik}) =$$

$$\begin{bmatrix} (u_{1k}^3 + u_{2k}^3 + u_{3k}^3 + 3x_{1k}^2 u_{1k} + 3x_{2k}^2 u_{2k} + 3x_{3k}^2 u_{3k} + \\ 3x_{1k} u_{1k}^2 + 3x_{2k} u_{2k}^2 + 3x_{3k} u_{3k}^2) \\ \begin{pmatrix} x_{1k}^2(u_{2k} + u_{3k}) + x_{2k}^2(u_{1k} + u_{3k}) + x_{3k}^2(u_{1k} + u_{2k}) \\ + (u_{1k}^2 + 2x_{1k} u_{1k})(x_{2k} + x_{3k} + u_{2k} + u_{3k}) \\ + (u_{2k}^2 + 2x_{2k} u_{2k})(x_{1k} + x_{3k} + u_{1k} + u_{3k}) \\ + (u_{3k}^2 + 2x_{3k} u_{3k})(x_{1k} + x_{2k} + u_{1k} + u_{2k}) \end{pmatrix} \\ (u_{1k} x_{2k} x_{3k} + u_{2k} x_{1k} x_{3k} + u_{3k} x_{1k} x_{2k} + x_{1k} u_{2k} u_{3k} + \\ x_{2k} u_{3k} u_{1k} + x_{3k} u_{1k} u_{2k} + u_{1k} u_{2k} u_{3k}) \end{bmatrix}$$

The $\overline{\eta}_k$ is calculated as

$$\overline{\eta}_k = \left[3\sigma^2(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1}) \quad 2\sigma^2(\hat{x}_{1k|k-1} + \hat{x}_{2k|k-1} + \hat{x}_{3k|k-1}) \quad 0\right]^T$$

The noise vector is taken as $\eta'_k = (\eta_k - \overline{\eta}_k)$ to make it zero mean. Now to implement any Gaussian filter the covariance of the noise sequence (R_k) may be calculated through $R_k = E[\eta'_k \eta'^T_k]$. From the expression of η'_k it is obvious that R_k would be 3x3 symmetrical matrix.

$R_k = [R_{11k} \quad R_{12k} \quad R_{13k}; \quad R_{21k} \quad R_{22k} \quad R_{23k}; \quad R_{31k} \quad R_{32k} \quad R_{33k}]$ Assuming that x_{ik} and u_{ik} are independent the different elements of R_k can be calculated as

$$R_{11k} = 9\sigma^2 E[\sum_{i=1}^{3} x_{ik}^4] + 45\sigma^4 E[\sum_{i=1}^{3} x_{ik}^2] + 18\sigma^4 E[\sum_{i=1}^{2} \sum_{j=i+1}^{3} x_{ik} x_{jk}] - 9\sigma^4 [\sum_{i=1}^{3} \hat{x}_{ik|k-1}]^2 + 45\sigma^6$$

$$R_{12k} = 6\sigma^2 E[\sum_{i=1}^{3} \sum_{\substack{j=1 \\ j \neq i}}^{3} x_{ik}^3 x_{jk}] + 6\sigma^2 E[\sum_{i=1}^{2} \sum_{j=i+1}^{3} x_{ik}^2 x_{jk}^2] + 36\sigma^4 E[\sum_{i=1}^{2} \sum_{j=i+1}^{3} x_{ik} x_{jk}] + 18\sigma^4 E[\sum_{i=1}^{3} x_{ik}^2] - 6\sigma^4 (\sum_{i=1}^{3} \hat{x}_{ik|k-1})[\sum_{i=1}^{3} \hat{x}_{ik|k-1}]^2 + 18\sigma^6$$

$$R_{13k} = 3\sigma^2 E[\sum_{i=1}^{3} \sum_{\substack{j=1 \\ j \neq i}}^{2} \sum_{\substack{l=j+1 \\ l \neq i}}^{3} x_{ik}^2 x_{jk} x_{lk}] + 3\sigma^4 E[\sum_{i=1}^{2} \sum_{j=i+1}^{3} x_{ik} x_{jk}]$$

$$R_{22k} = 2\sigma^2 E[\sum_{i=1}^{3} x_{ik}^4] + 10\sigma^2 E[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}^2 x_{jk}^2] + 16\sigma^2 E[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{2}\sum_{\substack{l=j+1\\ l\neq i}}^{3} x_{ik}^2 x_{jk} x_{lk}]$$
$$+ 4\sigma^2 E[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3} x_{ik}^3 x_{jk}] + 36\sigma^4 E[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik} x_{jk}] + 24\sigma^4 E[\sum_{i=1}^{3} x_{ik}^2]$$
$$- 4\sigma^4 [\sum_{i=1}^{3} \hat{x}_{ik|k-1}]^2 + 24\sigma^6$$

$$R_{23k} = \sigma^2 E[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3} x_{ik}^3 x_{jk}] + 4\sigma^2 E[\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{2}\sum_{\substack{l=j+1\\ l\neq i}}^{3} x_{ik}^2 x_{jk} x_{lk}] + 6\sigma^4 E[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik} x_{jk}]$$

$$R_{33k} = \sigma^2 E[\sum_{i=1}^{2}\sum_{j=i+1}^{3} x_{ik}^2 x_{jk}^2] + \sigma^4 E[\sum_{i=1}^{3} x_{ik}^2] + \sigma^6$$

Exact form of the R_k may be evaluated using the following relationships

$$E[x_{ik}^2] = \hat{x}_{ik|k-1}^2 + P_{iik}\ ;\ E[x_{ik}^4] = \hat{x}_{ik|k-1}^4 + 6\hat{x}_{ik|k-1}^2 P_{iik} + 3P_{iik}^2$$
$$E[x_{ik} x_{jk}] = \hat{x}_{ik|k-1}\hat{x}_{jk|k-1} + P_{ijk}$$
$$E[x_{ik}^2 x_{jk}^2] = (\hat{x}_{ik|k-1}^2 + P_{iik})(\hat{x}_{jk|k-1}^2 + P_{jjk}) + 2P_{ijk}^2 + 4\hat{x}_{ik|k-1}\hat{x}_{jk|k-1}P_{ijk}$$
$$E[x_{ik}^3 x_{jk}] = (\hat{x}_{ik|k-1}^3 + 3\hat{x}_{ik|k-1}P_{iik})\hat{x}_{jk|k-1} + 3P_{ijk}(\hat{x}_{ik|k-1}^2 + P_{iik})$$
$$E[x_{ik}^2 x_{jk} x_{lk}] = (\hat{x}_{ik|k-1}^2 + P_{iik})(\hat{x}_{jk|k-1}\hat{x}_{lk|k-1} + P_{jlk}) +$$
$$2\hat{x}_{ik|k-1}(\hat{x}_{jk|k-1}P_{ilk} + \hat{x}_{lk|k-1}P_{ijk}) + P_{ijk}P_{ilk}$$

where P is the prior error covariance matrix. During estimation the noise covariance is calculated using the above derived expression. Also the mathematical expression of (R_k) derived from the above formula has been verified using Monte Carlo run.

B. *Simulation Results:*

As the formulated problem is nonlinear in nature, nonlinear estimators have to be implemented. Although here the problem has been solved using extended Kalman filter (EKF) [16], more advanced nonlinear filters like unscented Kalman Filter (UKF) [17], quadrature based filters [18], central difference filter (CDF) [19], *etc* can also be implemented without any further calculation of noise parameters. The truth model as well as filter has been simulated in MATLAB environment. The truth is initialized as. $X_0 = [5\ \ 15\ \ 20\ \ 1.5\ \ -0.5\ \ -1]^T$. Sensor noise (u_{ik}) has been assumed to be white Gaussian with zero mean and covariance $\sigma^2 = diag[25\ \ 25\ \ 25]$. As the process noise covariance (Q_k) is zero, targets' velocities remain constant in their respective initial values during the simulation which has been carried out for 20 seconds with the sampling time 0.01 second. The estimated values of states have been initialized with $\hat{X}_0 = [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]^T$ and error covariance $P_0 = diag[9\ \ 25\ \ 64\ \ 2\ \ 0.25\ \ 16]$.

The truth and estimated values of positions obtained from homogeneous form of symmetric measurement equation for three particles for a single representative run have been plotted in figure1. Similar results are also obtained for other two forms of symmetric measurement equation and have not been shown here. The initial conditions are selected such a way that the targets cross each other within simulation interval. From simulation it has been found that the filter tracks the positions of the particles well but without identifying the particles (as shown in figure 1). It means that tracks are not labeled and exchange of tracks may occur during crossover.

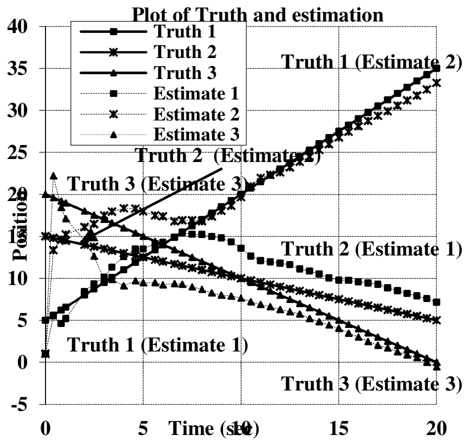

Figure 1 Truth and estimated values for a single representative run for homogeneous form of symmetric measurement equation

To label the track all the permutations of state have been considered and the estimated values of states are frozen for that permutation which has least sum of square error. In figure 2 and 3, root mean square error (RMSE) of second target obtained from EKF for 100 Monte Carlo runs have been compared for position and velocity respectively. It can be observed that in position plot there are kinks near 5th and 10th second which are due to the exchange of track labeling during crossover. From figure 2 and 3, it has also been observed that the RMSE of position and velocity is distinctly smaller in sum of product than sum of power form of symmetric measurement equation as also explored in [15]; whereas performance of homogeneous form of symmetric measurement equation is comparable to that of sum of product form and sometimes it is better. Similar results have been obtained for other particles and have not been included here.

V. DISCUSSIONS AND CONCLUSION

In this work a new form of symmetric measurement equation is proposed from homogeneous symmetric function. The noise vector and its covariance have been calculated for any number of particles in motion. The observability condition for homogeneous symmetric measurement has been derived. A case study of three particles in motion has been discussed and targets' positions and velocities have been estimated using extended Kalman filter for three forms of symmetric measurement namely sum of power, sum of product and homogeneous symmetry. The results among the three forms of symmetric measurement have been compared in terms of RMSE. Performance of estimator with homogeneous symmetric measurement equation has been obtained better than the other two forms.

As the noise statistics have been calculated mathematically, other types of advanced Gaussian filter such as unscented Kalman filter (UKF), Gauss Hermite filter (GHF) or central difference filter (CDF) *etc* may be implemented easily for better estimation of states. Also evaluation of performance for other forms of symmetric measurement equation derived from various symmetric functions (viz. complete homogeneous symmetric functions, Schur functions of different orders) remains under the scope of future work. The proposed homogeneous form would become a promising candidate in family of symmetric measurement equations used for tracking of multiple targets to overcome data association problem.

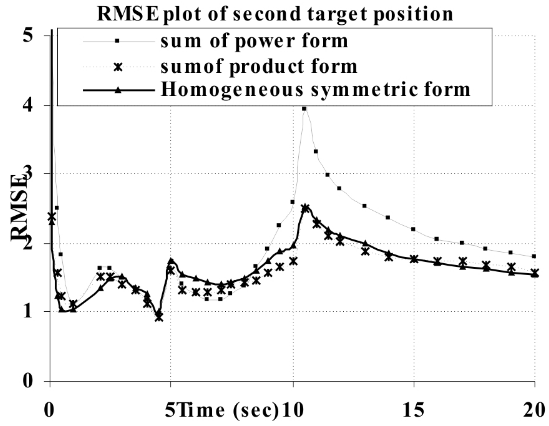

Fig 2: RMSE of position for different forms of symmetric measurement equation

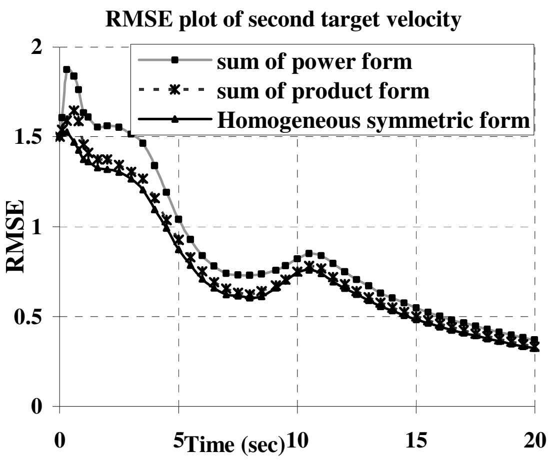

Fig. 3: RMSE of velocity for different forms of symmetric measurement equation

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