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Risk-sensitive formulation of unscented Kalman filter

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Abstract: A novel method for non-linear risk-sensitive estimation based on the unscented transform has been developed. The proposed filter, referred to as risk-sensitive unscented Kalman filter (RSUKF), and would be able to overcome inherent disadvantages associated with the earlier reported extended risk-sensitive filter (ERSF). The theory and formulation of RSUKF has been presented and possible variants thereof indicated. Using two well-known non-linear examples, the superiority of RSUKF performance has been demonstrated. As the RSUKF has similar computational efficiency and better robustness as compared with the ERSF, the former may be more suitable for onboard implementation, quick exploration of risk-sensitive filtering for non-linear problems and also for generating proposal densities for more computation intensive risk-sensitive particle filters.

1 Introduction

Risk-sensitive estimator [1–3] that originates from the risk-sensitive control law, has increased robustness compared with risk-neutral filters. The said estimator optimises (minimises) an exponential cost criterion. For linear Gaussian signal models, a closed-form solution exists and the risk-sensitive filter (RSF) has been formulated as Kalman filter-like recursions [1]. In linear Gaussian models, the RSF algorithm is closely related to the H_∞ filter [1]. However, difficulty arises in non-linear and/or non-Gaussian systems as the associated integrals become intractable. For non-linear risk-sensitive estimation, an approximate version, the extended risk-sensitive filter (ERSF), based on the extended Kalman filter (EKF) has been proposed in literature [4]. However, the limitations of the EKF, including smoothness requirement for the functions are inherited by the ERSF. Hence, the ERSF fails to provide a good estimate and often diverges for severely nonlinear signal models.

To overcome the limitations of the ERSF several methods have been proposed earlier by the present authors. These are the risk-sensitive particle filter (RSPF) [5–7], the adaptive grid risk-sensitive filter (AGRSF) [8, 9] and the central difference risk-sensitive filter (CDRSF) [10]. Among these non-linear RSFs, in RSPF and AGRSF intractable integrals are evaluated using numerical techniques based on particle filtering and adaptive grid-based filtering respectively. As a result the computational load is very high for these two filters and may not be suitable in many real-time onboard applications. The CDRSF, on the other hand, is computationally efficient compared with RSPF and AGRSF.

In this paper a novel technique for non-linear RSF, based on unscented Kalman filter (UKF), has been proposed. In the proposed filtering technique the intractable integrals have been approximately evaluated using finite number of deterministic support points (called sigma points) and applying unscented transform [11, 12]. Accordingly, the proposed filter is named as risk-sensitive unscented Kalman filter (RSUKF).

It may be noted that both the CDRSF and the UKF (on which the proposed RSF is based) belong to the class ‘sigma point’ approximate filters. However, the unscented transform (UT)-based approach provides greater flexibility as discussed later in this paper.

The rest of this paper is organised as follows. The RSF formulation according to [1] is briefly discussed in the next section. This is followed by the theory of formulating the RSUKF filter for the prior and posterior estimations and a presentation of the algorithms for the general non-linear case and some special cases. The proposed method has then been applied to two non-linear problems available in literature and the results are compared with the ERSF. Some pragmatics and general comments are provided in the discussion section, which is followed by concluding comments.

2 RSF

Let us consider the non-linear plant described by the state and measurement equations as follows:

$$x_{k+1} = \phi(x_k) + w_k \tag{1}$$

$$y_k = \gamma(x_k) + v_k \tag{2}$$

where $x_k \in R^n$ denotes the state of the system, $y_k \in R^p$ denotes the measurement at the instant k, where $k = \{0, 1, 2, 3, \ldots, N\}$, $\phi(x_k)$ and $\gamma(x_k)$ are known non-linear functions of x_k and k. The process noise $w_k \in R^n$ and measurement noise $v_k \in R^p$ are assumed to be uncorrelated and normally distributed with covariance Q and R respectively.

The following notations have been used for the probability density functions:

$$f(x_{k+1}|x_k) \triangleq p_{X_{k+1}|X_k}(.|x_k) \quad \text{and} \quad g(y_k|x_k) \triangleq p_{Y_k|X_k}(.|x_k)$$

The objective is to estimate a known function $\Phi(x)$ of the state variables. The estimate is designated as $\hat{\Phi}(x)$ and its optimal value in the risk-sensitive sense is denoted by $\hat{\Phi}^*(x)$ which minimises the cost function

$$C\left(\hat{\Phi}_1, \ldots, \hat{\Phi}_k\right) = E\left[\exp\left(\mu_1 \sum_{i=1}^{k-1} \rho_1\left(\Phi(x_i) - \hat{\Phi}_i^*\right)\right) + \left(\mu_2 \rho_2\left(\Phi(x_k) - \hat{\Phi}_k^*\right)\right)\right]$$

where $\mu_1 \geq 0$ and $\mu_2 > 0$ are two risk parameters. Functions $\rho_1(.)$ and $\rho_2(.)$ are both strictly convex, continuous and bounded from below, attaining global minima at 0. Conventionally they represent squared deviation for the mean squared measure.

In particular, the minimum risk-sensitive estimate (MRSE) is defined by

$$\hat{\Phi}_k^* = \arg\min C\left(\hat{\Phi}_1^*, \ldots, \hat{\Phi}_{k-1}^*, \hat{\Phi}_k\right) \tag{3}$$

It can be shown that the MRSE satisfies the following recursion

$$\sigma_{k+1}(x_{k+1}) = \int f(x_{k+1}|x_k) g(y_k|x_k) \exp(\mu_1 \rho_1(\Phi(x_k) - \hat{\Phi}_k^*)) \sigma_k(x_k) \mathrm{d}x \tag{4}$$

$$\hat{\Phi}_k^* = \arg\min_{\alpha \in R} \int \exp\left(\mu_2 \rho_2\left(\Phi\left(x_k\right) - \alpha\right)\right) \sigma_k\left(x_k\right) \mathrm{d}x_k \tag{5}$$

where $\sigma_k(x_k)$ represents an information state [1] and may be normalised and α is a parameter.

Here, the variables to be estimated are the state variables themselves ($\Phi(x) = x$), and both the convex functions $\rho_1(.)$ and $\rho_2(.)$ are known quadratic functions of vectors, of the form, $\rho_j(\alpha) = \alpha^{\mathrm{T}} \alpha$ for $j = 1, 2$.

In particular, for prior estimation, we denote the optimal estimate as $\hat{x}_{k+1|k}$ and accordingly we change the notation for the information state.

Hence, (4) and (5) can be rewritten as

$$\sigma_{k+1|k}(x_{k+1}) = \int f(x_{k+1}|x_k) g(y_k|x_k) \times \exp\left(\mu_1 (x_k - \hat{x}_{k|k-1})^{\mathrm{T}} (x_k - \hat{x}_{k|k-1})\right) \times \sigma_{k|k-1}(x_k) \mathrm{d}x_k \tag{6}$$

and

$$\hat{x}_{k|k-1} = \arg\min_{\alpha \in R} \int \exp\left(\mu_2 \left(x_k - \alpha\right)^{\mathrm{T}} \left(x_k - \alpha\right)\right) \sigma_{k|k-1}\left(x_k\right) \mathrm{d}x_k \tag{7}$$

3 Formulation of RSUKF

A general, closed form evaluation of the integral in (6) is intractable and hence, the well-known UT-based approach has been used to obtain the approximate solution of the said integral.

However, the UT is not directly applicable and additional steps and assumptions are needed. These are required for (i) obtaining the optimal risk-sensitive (RS) estimate from (7), (ii) reformulating the RS recursive equations (6) and (7) and (iii) modifying the covariance to take care of the risk factor.

First, the information state $\sigma_k(x_k)$ is assumed to be normalised so that it may be interpreted as a probability density distribution. This allows a probabilistic interpretation of the concerned equations and makes the treatment amenable to UT. The algorithm, however, is so arranged that explicit normalisation is avoided.

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In risk-neutral estimation, the mean squared error is to be minimised and the mean of the underlying probability distribution is the solution. The definition of the optimal RS estimate given in (7) does not automatically provide the mean as the optimal estimate. However, the mean is the optimal estimate under assumption of Gaussianity. We can further state that even when $\sigma_k(x_k)$ is non-Gaussian but may be represented by a mixture of Gaussian densities of the form $\sigma_k(x_k) = \sum_{i=1}^{n} w_i \mathcal{N}(m_i, P_i)$, (where w_i is a weight, m_i and P_i are respectively the mean and covariance of the component densities and n the number of such component densities) the expected value of the distribution $\sigma_k(x_k)$ is the optimal RS estimate, for at least the following cases:

(i) when the component densities have the same mean (but different covariances),

(ii) when the distribution is composed of identical pairs of mixtures, where each of such identical pair may have components described in (i) above.

Note that the above two situations cover a wide variety of symmetric distributions, including multimodal distributions.

The UKF algorithm makes an implicit Gaussian random variable (GRV) assumption [13] although maintaining the mean and covariance through non-linear transformations. The particular method by which the risk-sensitivity is accommodated within the UKF formalism is described next.

3.1 The prior form of RSUKF

In order that the UT may be applied, the integrand (6) is first decomposed into the following expressions. The integrand (6) is rewritten as

$$\sigma_{k+1|k}(x_{k+1}) = \int f(x_{k+1} \mid x_k)\sigma_{k|k}(x_k)dx_k \tag{8}$$

where

$$\sigma_{k|k}(x_k) = g(y_k \mid x_k)\exp(\mu_1(x_k - \hat{x}_{k|k-1})^T \times (x_k - \hat{x}_{k|k-1}))\sigma_{k|k-1}(x_k) \tag{9}$$

Let us define $\sigma^{+}_{k|k-1}(x_k)$ as

$$\sigma^{+}_{k|k-1}(x_k) = \exp(\mu_1(x_k - \hat{x}_{k|k-1})^T \times (x_k - \hat{x}_{k|k-1}))\sigma_{k|k-1}(x_k) \tag{10}$$

Then (9) can be rewritten as

$$\sigma_{k|k}(x_k) = g(y_k \mid x_k)\sigma^{+}_{k|k-1}(x_k) \tag{11}$$

The expression (8) provides the basis for time-update or predictor step equation. The optimal estimate is then obtained by using (7). The expression (10) defines the 'Risk-update' step. It is specific to RSF and there is no analogue in the risk-neutral case. Expression (11) provides the modified measurement update or corrector step equation.

The computing steps are broadly as follows:

(i) Initialisation of filter with $\bar{\sigma}_{0|0}$ and $P_{0|0}$ and calculation of the weights of the sigma points.

(ii) The time update step or predictor step starts with the knowledge of $\bar{\sigma}_{k|k}$ and $P_{k|k}$ from the previous cycle (or from the initial conditions, for the first cycle) and computation of the corresponding sigma points. The step is identical to UKF as the governing relations are also the same, which involves transforming the sigma points and adding the effect of process noise. From the transformed sigma points the updated information state $\sigma_{k+1|k}(x_{k+1})$ specified by its mean and covariance $\bar{\sigma}_{k+1|k}$ and $P_{k+1|k}$ may be computed using the weighted sum technique of UT.

(iii) The optimal estimate may be replaced by the mean of $\sigma_{k+1|k}(x_{k+1})$ under Gaussianity assumption, as discussed in the previous subsection.

(iv) In 'Risk-update' step, the term $\exp(\mu_1(x_k - \hat{x}_{k|k-1})^T (x_k - \hat{x}_{k|k-1}))\sigma_{k|k-1}(x_k)$ is computed by the Gaussian distribution assumption, whereby, only the covariance term is affected (the mean of both terms being the same and the resultant covariance is expressed as $P^{+}_{k+1|k} = (P^{-1}_{k+1|k} - 2\mu I)^{-1}$).

(v) As the mean and covariance undergo changes because of the time update and risk update, it is desirable that the sigma points be recalculated on the basis of $\bar{\sigma}_{k+1|k}(x_{k+1})$ and $P^{+}_{k+1|k}$. If the changes are not significant, this step may be skipped to save some computation.

(vi) The measurement update step is identical in form to the standard UKF, yielding $\bar{\sigma}_{k|k}$ and $P_{k|k}$ using the weighted sum technique of UT.

The specifics of the proposed algorithm are described in the Section 4.

3.2 The posterior form of RSUKF

The posterior form of the risk-sensitive estimate is given by

$$\hat{x}_{k|k} = \underset{\zeta \in R^n}{\arg\min} \int \exp\left(\mu_2(x_k - \alpha)^T (x_k - \alpha)\right) \sigma_{k|k}(x_k)dx_k \tag{12}$$

where

$$\sigma_{k|k}(x_k) = \int f(x_k \mid x_{k-1})g(y_k \mid x_k)\exp(\mu_1(x_{k-1} - \hat{x}_{k-1|k-1})^T \times (x_{k-1} - \hat{x}_{k-1|k-1}))\sigma_{k-1|k-1}(x_{k-1})dx_{k-1} \tag{13}$$

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In a manner analogous to the prior RS estimate, we introduce the distribution

$$\sigma^{+}_{k-1|k-1} = \exp(\mu_1 (x_{k-1} - \hat{x}_{k-1|k-1})^{\mathrm{T}} (x_{k-1} - \hat{x}_{k-1|k-1})) \times \sigma_{k-1|k-1}(x_{k-1}) \quad (14)$$

Hence,

$$\sigma_{k|k-1}(x_{k-1}) = \int f(x_k \mid x_{k-1}) \sigma^{+}_{k-1|k-1}(x_{k-1}) \, \mathrm{d}x_{k-1} \quad (15)$$

where

$$\sigma_{k|k}(x_k) = g(y_k \mid x_k) \, \sigma_{k|k-1}(x_{k-1}) \quad (16)$$

Equation (14) is defined as the risk-sensitive update step, (15) corresponds to the predictor step, and (16) corresponds to the corrector (measurement update) step.

The filter algorithm for posterior estimation is not being included in this paper but the steps for the posterior RSUKF proceed in a manner analogous to those for the prior form. However, in the posterior form, it is convenient to consider the risk-sensitive term in the prediction step, and consequently, appropriate changes must be incorporated in the filter algorithm in accordance with (14)–(16).

4 RSUKF algorithm for prior estimation

4.1 General RSUKF algorithm for non-linear systems

Step (i) Filter initialisation:

$$\text{Initialise the filter with } \bar{\sigma}_{0|0} = \hat{x}_{0|0} \text{ and } P_{0|0} \quad (17)$$

- Calculate weights

$$W_0 = \kappa/(n+\kappa), \; W_i = 1/2(n+\kappa), \; W_{i+n} = 1/2(n+\kappa) \quad (18)$$

Step (ii) Predictor step:

- Assign sigma points with $\bar{\sigma}_{k|k}$ and $P_{k|k}$ as

$$\chi_k = [\bar{\sigma}_{k|k} \; \bar{\sigma}_{k|k} + (\sqrt{(n+\kappa)P_{k|k}})_i \quad \bar{\sigma}_{k|k} - (\sqrt{(n+\kappa)P_{k|k}})_i] \quad (19)$$

where $(\sqrt{(n+\kappa)P_{k|k}})_i$ is the ith column of the matrix square root of $(n+\kappa)P_{xx}$, where the matrix square root A of P is of the form $P = AA^{\mathrm{T}}$, [12].

- Update sigma points using process dynamics

$$\chi_{i,k+1|k} = \phi(\chi_{i,k}) \quad (20)$$

- Compute time updated mean and covariance

$$\bar{\sigma}_{k+1|k} = \sum_{i=0}^{2n} W_i \chi_{i,k+1|k} \quad (21)$$

$$P_{k+1|k} = \sum_{i=0}^{2n} W_i [\chi_{i,k+1|k} - \bar{\sigma}_{k+1|k}][\chi_{i,k+1|k} - \bar{\sigma}_{k+1|k}]^T + Q \quad (22)$$

Step (iii) Optimal RS estimate:

- With Gaussian approximation, the optimal estimate is

$$\hat{x}_{k+1|k} = \bar{\sigma}_{k+1|k} \quad (23)$$

Step (iv) Risk-sensitive update:

- The mean remains the same and the covariance is updated with

$$P^{+}_{k+1|k} = (P^{-1}_{k+1|k} - 2\mu_1 I)^{-1} \quad (24)$$

Step (v) Recompute sigma points:

- Sigma points are recomputed with mean as $\bar{\sigma}_{k+1|k}(x_{k+1})$ and covariance as $P^{+}_{k+1|k}$

Step (vi) Corrector step:

- Projected measurement at each sigma point

$$Y_{i,k+1|k} = \gamma(\chi_{i,k+1|k}) \quad (25)$$

- Mean of measurement

$$\hat{y}^{-}_{k+1} = \sum_{i=0}^{2n} W_i Y_{i,k+1|k} \quad (26)$$

- Calculation of covariances

$$P_{y_{k+1}y_{k+1}} = \sum_{i=0}^{2n} W_i [Y_{i,k+1|k} - \hat{y}^{-}_{k+1}][Y_{i,k+1|k} - \hat{y}^{-}_{k+1}]^T + R \quad (27)$$

$$P_{x_{k+1}y_{k+1}} = \sum_{i=0}^{2n} W_i [\chi_{i,k+1|k} - \hat{x}_{k+1|k}][Y_{i,k+1|k} - \hat{y}^{-}_{k+1}]^T \quad (28)$$

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- Calculation of Kalman gain

$$K_{k+1} = P_{x_{k+1 k+1}} P^{-1}_{y_{k+1 k+1}} \quad (29)$$

- Posterior state values

$$x_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}^{-}_{k+1}) \quad (30)$$

- Posterior error covariance matrix is given by

$$P_{k+1|k+1} = P^{+}_{k+1|k} - K_{k+1} P_{y_{k+1} y_{k+1}} K^{T}_{k+1} \quad (31)$$

4.2 Special cases

For a non-linear signal model with either a linear plant or linear measurements, the RSUKF algorithm can be simplified, leading to significant reduction of computation time, especially for systems with high dimensions. This can be done by using the RSUKF algorithm for the step corresponding to the non-linear equation and the Kalman filter algorithm for the step corresponding to the linear equation. However, there are specific considerations which must be taken into account in each case. Hence, the modifications to the general RSUKF algorithm (in Section 4.1) in each of these two special cases are being stated below.

4.2.1 Non-linear system and linear measurement:

In this special case, the algorithm may be modified as follows:

Steps (*i*) – (*iv*) would be the same as the general RSUKF algorithm stated in Section 4.1.

Step (*v*) is skipped.

Step (*vi*) the *corrector step* would be the same as the corrector step of the Kalman filter.

4.2.2 Linear system and non-linear measurement:

For linear process equation and non-linear measurement equation, the predictor step would be computed by using the Kalman filter algorithm and the remaining steps are the same as that in the general RSUKF algorithm stated in Section 4.1.

In particular,

Step (i) *filter initialisation* is done in the same way as the general RSUKF algorithm.

Step (ii) the *predictor step* is computed in a manner similar to the predictor step of a Kalman filter.

Steps (iii) – (vi) are the same as the general RSUKF algorithm in Section 4.1.

5 Case study

Two examples have been provided in this section to compare the performance of the RSUKF and the ERSF. In both the examples, numerical value of n and κ has been chosen as discussed by Julier *et al.* [12] where heuristically for Gaussian noise value for $(n + \kappa)$ was taken as 3.

5.1 Example 1

This example uses a plant where both the state model and the measurement model are severely non-linear. The plant model inspired by [13] has strong non-linearity, with one unstable and two stable equilibrium points. A brief description of the plant is provided below and a detailed problem statement is available in [10]. The process and measurement equations are given respectively by

$$x_{k+1} = \phi(x_k) + w_k \quad \text{where } \phi(x) = x + \Delta t 5x(1 - x^2),$$
$$\times\, w_k \sim \mathcal{N}(0, b^2 \Delta t) \quad (32)$$

And

$$y_k = \gamma(x_k) + v_k \quad \text{where } \gamma(x) = \Delta t x(1 - 0.5x),$$
$$v_k \sim \mathcal{N}(0, d^2 \Delta t) \quad (33)$$

Value of $\Delta t = 0.01$ s, $x_0 = -0.2$, $\hat{x}_{0|0} = 0.8$, $P_{0|0} = 2$, $b = 0.5$, $d = 0.1$. We have considered the time span from 0 to 0.8 s. Risk-sensitive parameter μ_1 has been chosen as 0.0756.

This problem allows easy detection of deficient state estimators as moderate estimation errors force the estimate to settle down at the wrong equilibrium point, leading to an easily discernable track loss.

Results:

- Tracking performance for ERSF and proposed RSUKF has been shown for a typical run in Fig. 1. For this

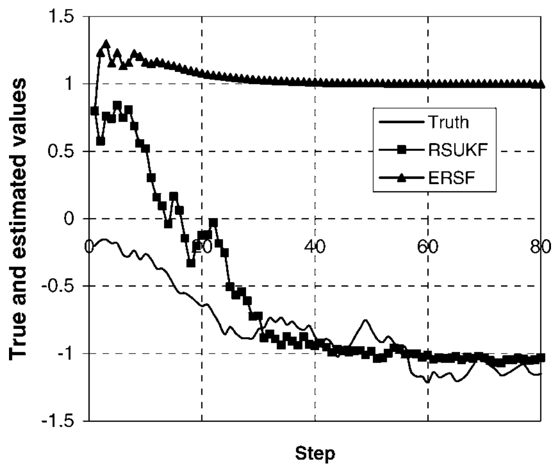

Figure 1 *Comparison of ERSF and RSUKF performance for a representative run*

representative run, it has been observed that the ERSF fails to track whereas the performance of the RSUKF is satisfactory.

- The performance of the two filters is compared first in terms of percentage of fail count, which has been defined as the number of cases out of 100 (in a population of 1000 Monte Carlo run) where a filter fails to track the truth i.e. the filter settles at the wrong equilibrium point. For ERSF the percentage fail count is about 26% whereas for RSUKF it is about 1%. The numbers indicate the improvement of estimation accuracy with the proposed filter in comparison to the traditional ERSF.

- Comparison of filter performance has also been carried out with root mean square error (RMSE). The RMSE of RSUKF and ERSF for 1000 Monte Carlo runs are compared in Fig. 2. The figure shows that RMSE of ERSF is high (because of a large population of track loss cases), whereas the RMSE of RSUKF settles at 0.2 units.

5.2 Example 2

This example considers the well-known two-dimensional bearing only tracking (BOT) problem [14, 15] with a linear system and highly non-linear measurements. The BOT problem, in which bearings information is processed to evaluate the position and velocity estimate of the target, has attracted the attention of many workers because of its practical military and civil applications including underwater weapon systems, IR seeker-based tracking, sonar-based robotic navigation and TV camera or stereo microphone-based people tracking. This problem has been chosen as it is a highly non-linear problem and has poor observability. Thus this problem can act as a benchmark to demonstrate the efficiency of filtering techniques.

The target and a tracker, mounted on a moving platform, are moving parallelly in the same direction (x-axis) with constant velocity. The target kinematics and the tracking platform kinematics are shown in Fig. 3. A more detailed problem description can be found in [14, 15].

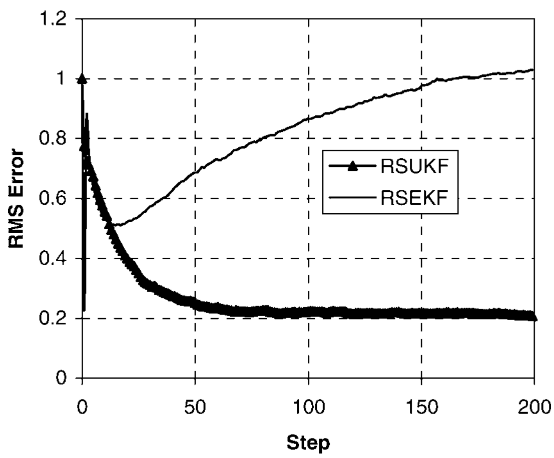

Figure 2 *Root mean square error plot of ERSF and RSUKF for 1000 runs*

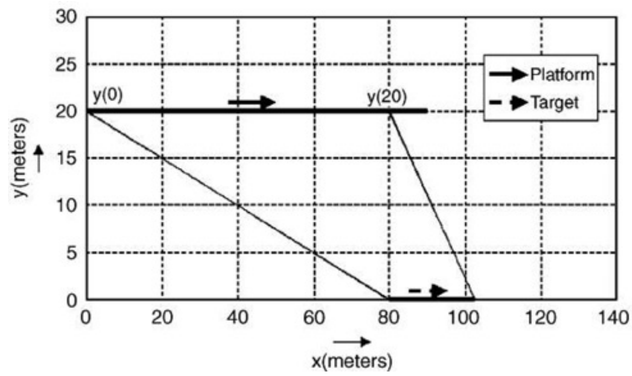

Figure 3 *Two-dimensional bearing only tracking problem scenario*

The kinematics of platform is given in discrete-form as

$$x_k^p = \bar{x}_k^p + \Delta x_k^p \tag{34}$$

$$y_k^p = \bar{y}_k^p + \Delta y_k^p, \quad k = 0, 1, \ldots, N \tag{35}$$

where $\bar{x}_k^p$ (superscript 'p' denotes platform) and $\bar{y}_k^p$ are average platform position coordinates and Δx_k^p and Δy_k^p are the mutually independent zero mean Gaussian white noise sequences with variances $r_x = 1\ \text{m}^2$ and $r_y = 1\ \text{m}^2$, respectively. The mean position of platform in the coordinate system is given by

$$\bar{x}_k^p = 4k\tau \tag{36}$$

and

$$\bar{y}_k^p = 20 \tag{37}$$

Considering the position along the x-axis in meters and the velocity in m/s as the two states of the system, the target kinematics can be written in discrete-time state space formulation as follows

$$x_{k+1} = F_k x_k + G_k w_k \tag{38}$$

where $F_k = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$, $G_k = \begin{bmatrix} \tau^2/2 \\ \tau \end{bmatrix}$ and w_k is independent zero mean Gaussian white acceleration noise with variance $q = 0.01\ \text{m}^2/\text{s}^4$. The sampling time, τ, is 1 s.

The initial values of the true states are given by

$$x_0 = \begin{bmatrix} 80 \\ 1 \end{bmatrix}$$

For RSF, a value of 0.0001 has been chosen for risk-sensitive parameter (μ_1).

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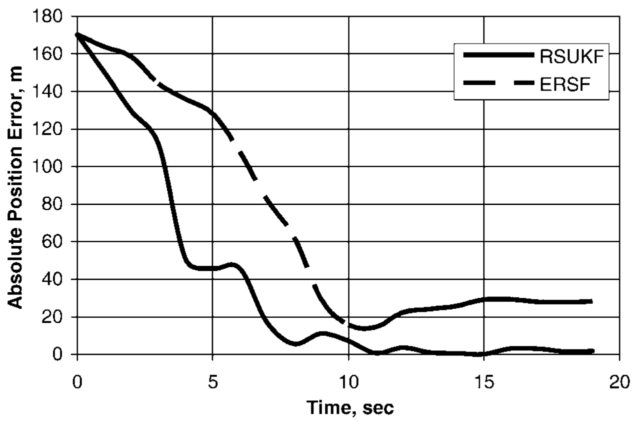

Figure 4 *Absolute value of position error of ERKF and RSUKF in case of large initial position error*

The absolute error in position estimates obtained from the ERSF and the RSUKF have been plotted in Fig. 4 for a single representative run. The initial position error has been assumed to be very high for this run so that robustness of the ERSF and RSUKF can be tested.

Results: The improvement of performance by using RSUKF over ERSF is evident. The RSUKF is much more robust with respect to large initial errors. In fact, a 1000 cycle MC run shows that there are over 25 cases of track loss in ERSF. The track loss with RSUKF for the particular value of the risk factor is about a seventh of this number of cases.

6 Discussions

• Results presented here and earlier publications show that ERSF inherits the shortcomings of EKF, whereas RSUKF retains the advantage of UKF in handling non-linear models. This may be attributed to the fact that the UKF considers higher-order terms in the Taylor series whereas a simple EKF makes only a first-order approximation.

• In RSF risk-sensitive parameter provides a tool for design trade-off between nominal and robust filtering performance.

• The algorithm presented here uses a basic form of UKF proposed by Julier *et al.* [12]. Various improved versions of the UKF (such as scaled UKF [16] and square root UKF [11]) exist in literature. Risk-sensitive formulation of those variants of UKF can also be done in a similar manner. In some rare cases, RSUKF steps may yield a singular covariance matrix. The scaled version of UT and the square root implementation are recommended for these cases.

• This filter has been found to be computationally as efficient as the CDRSF and the performance of the filter is similar to the CDRSF. However, an advantage of the proposed filter over the CDRSF is that in the scaled variant [16] of UT, more design freedom is available.

• The condition $(P_{k+1|k}^{-1} - 2\mu_1 I) > 0$ should be satisfied for positive covariance matrix. Hence, the risk-sensitive factor μ_1 should be assumed to be low enough to ensure convergence of the RSUKF. It is to be noted, however, that the ERSF and CDRSF too suffer from a similar kind of restriction.

• Although the RSPF and AGRSF have better performance than the CDRSF and RSUKF, the computational load is very high for these two filters and these filters may not be suitable in many real-time applications, where RSUKF will be a better candidate, especially for higher-order systems.

• For getting more accurate result with a given number of particles, RSPF requires a good proposal density. So far use of ERSF was proposed for this role [7]. Clearly, RSUKF will be much more suitable.

• Although we used the result that optimal RS estimate is the mean with a Gaussian distribution assumption, it may be shown that this is true for many other symmetrical distributions using the Gaussian mixture models.

7 Conclusion

An UT-based solution of non-linear risk-sensitive estimation problem has been proposed and its theory and algorithm for RSUKF have been developed. The superiority of RSUKF over ERSF has been demonstrated with examples. In particular, the RSUKF was found to be maintaining track in a large number of cases where ERSF lost track. Performance of the proposed filter was found to be comparable to the earlier reported CDRSF but the RSUKF has more flexibility. Although RSUKF is expected to be less accurate than computationally intensive AGRSF and RSPF, the numerical efficiency of the former filter (as also the CDRSF) would enable quick exploration of possible advantages of the RSF approach for higher-dimensioned non-linear problems. Owing to numerical efficiency and expected robustness to modelling errors, the proposed filter may become a candidate for onboard applications.

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