On observability of irregular descriptor systems ⋆

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Abstract: In this paper, a method is proposed to check the observability of irregular linear time invariant descriptor systems. The method is based on the properties of restricted system equivalent, derived here from a given descriptor system by means of simple matrix theory. Using restricted system equivalent form, equivalence between the observability of a given descriptor system and that of its corresponding normal system has been established.

Keywords: Linear Descriptor Systems, Irregular systems, Observability.

1. INTRODUCTION

In the last three decades, considerable amount of research was focused on the analysis, design, and numerical simulation techniques for descriptor systems, which arise in modeling of many real and practical systems, e.g. electrical network analysis, power systems, constrained mechanics, economic systems, chemical process control, see, Campbell (1980, 1982); Dai (1989a); Berman et al. (1996); Duan (2010); Luenberger (1979); Lewis (1986). Depending on the area, descriptor systems are referred by verity of names, *viz.* differential algebraic equations (DAEs), singular, implicit, generalized state space, noncanonic, degenerate, semi-state and nonstandard systems. In this paper, we consider the following linear time invariant descriptor system

$$\tilde{E}\dot{x} = \tilde{A}x + \tilde{B}u, \tag{1a}$$

$$\tilde{y} = \tilde{C}x, \tag{1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, $\tilde{y} \in \mathbb{R}^p$, are the state vector, the input vector and the output vector, respectively. $\tilde{E} \in \mathbb{R}^{m\times n}$, $\tilde{A} \in \mathbb{R}^{m\times n}$, $\tilde{B} \in \mathbb{R}^{m\times k}$, and $\tilde{C} \in \mathbb{R}^{p\times n}$ are known constant matrices. Where the $\text{rank}(\tilde{E}) = n_0$. If $m = n$ and $\tilde{E}$ is nonsingular, then the system (1) is called normal system. If $m = n$ and $\exists\ \lambda \in \mathbb{C}$ such that $\det(\lambda\tilde{E} - \tilde{A}) \neq 0$, then the system (1) is called regular descriptor system.

Rosenbrock (1974) and Verghese et al. (1981) have developed frequency-domain methods for various concepts of linear regular descriptor systems. Yip and Sincovec (1981) extended such concepts in time-domain. Cobb (1984) and Christodoulou and Paraskevopoulos (1985) integrated and completed the above theories for solvability, controllability, and observability properties of regular systems. Dai (1989b) has extended the concepts of reachability, controllability, and observability from regular systems to irregular cases. Ishihara and Terra (2001) have established a fairly extensive theory of impulse (I-) observability of irregular systems. In this paper, under the condition of impulse observability, equivalence between the reachable (R-) observability of irregular system and observability of its corresponding normal system is established.

Our approach is as follows: Under the assumption of I-observability of the given descriptor system, a restricted system equivalent is derived by means of matrix transformation theory, which satisfies rank conditions as given in the equations (3) and (5). Then, Theorem 3 establishes equivalence between R-observability of given descriptor system and observability of corresponding normal system.

2. OBSERVABILITY OF DESCRIPTOR SYSTEMS

Let us make the following conditions on the system (1)

(H1) $\text{rank}\begin{bmatrix} \tilde{E} & \tilde{A} \\ 0 & \tilde{E} \\ 0 & \tilde{C} \end{bmatrix} = n + \text{rank}(\tilde{E})$,

(H2) $\text{rank}\begin{bmatrix} \lambda\tilde{E} - \tilde{A} \\ \tilde{C} \end{bmatrix} = n\ \forall\ \lambda\ \in\ \mathbb{C}$, where $\mathbb{C}$ represents the set of complex numbers.

Some useful terms are defined as follows Dai (1989b):

(1) System (1) is said to be *strongly observable* if both of the conditions $(H1)$ and $(H2)$ hold.
(2) System (1) is called *I-observable* if condition $(H1)$ holds.
(3) System (1) is said to be *R-observable* if condition $(H2)$ holds.

Since rank $\tilde{E} = n_0$, there exists a nonsingular matrix Q such that $Q\tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $Q\tilde{A} = \begin{bmatrix} A \\ A_1 \end{bmatrix}$, $Q\tilde{B} = \begin{bmatrix} B \\ B_1 \end{bmatrix}$ and system (1) is restricted system equivalent, see Darouach and Boutayeb (1995), to the following system.

$$E\dot{x} = Ax + Bu, \tag{2a}$$

$$y = Cx, \tag{2b}$$

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where $E \in \mathbb{R}^{n_0\times n}$ (full row rank), $A \in \mathbb{R}^{n_0\times n}$, $B \in \mathbb{R}^{n_0\times k}$, $A_1 \in \mathbb{R}^{(m-n_0)\times n}$, $B_1 \in \mathbb{R}^{(m-n_0)\times k}$, $y = \begin{bmatrix}-B_1u\\ \tilde{y}\end{bmatrix} \in \mathbb{R}^t$, $C = \begin{bmatrix}A_1\\ \tilde{C}\end{bmatrix} \in \mathbb{R}^{t\times n}$, and $t = m+p-n_0$.

Now, if the system (1) satisfies condition (H1), then

$$\operatorname{rank}\begin{bmatrix}E & A\\ 0 & E\\ 0 & C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\tilde{E} & \tilde{A}\\ 0 & \tilde{E}\\ 0 & \tilde{C}\end{bmatrix} = n+n_0$$

$$\Rightarrow \operatorname{rank}\begin{bmatrix}E\\ C\end{bmatrix} = n. \tag{3}$$

Also, if the system (1) satisfies condition (H2) then

$$\operatorname{rank}\begin{bmatrix}\lambda E-A\\ C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\lambda\tilde{E}-\tilde{A}\\ \tilde{C}\end{bmatrix} = n \;\forall\; \lambda \in \mathbb{C} \tag{4}$$

3. MAIN RESULTS

Theorem 1. Suppose the equation (3) holds for the system (2). Then there exists a full column rank matrix $R \in \mathbb{R}^{n\times n_0}$ such that

$$\operatorname{rank}\begin{bmatrix}I_n-RE\\ C\end{bmatrix} = \operatorname{rank}(C). \tag{5}$$

Proof. Let rank $C = p_1$, $\exists$ a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that $CP = [S\;\; 0_{t\times n-p_1}]$. Where S is full column rank matrix.
Let $EP = [E_1\;\; E_2]$.
Thus

$$\operatorname{rank}\begin{bmatrix}E\\ C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}EP\\ CP\end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix}E_1 & E_2\\ S & 0\end{bmatrix} = n. \tag{6}$$

It follows from (6) that
$\operatorname{rank}[E_2] = n-p_1$ (full column rank).

So $\exists$ a nonsingular matrix $R_0 \in \mathbb{R}^{n_0\times n_0}$ such that $\begin{bmatrix}0_{(n-n_0)\times n_0}\\ R_0\end{bmatrix}[E_2] = \begin{bmatrix}0_{p_1\times(n-p_1)}\\ I_{n-p_1}\end{bmatrix}$.
Let $\begin{bmatrix}0_{(n-n_0)\times n_0}\\ R_0\end{bmatrix}[E_1] = \begin{bmatrix}E_{11}\\ E_{21}\end{bmatrix}$. Where $E_{11} \in \mathbb{R}^{p_1\times p_1}$ and $E_{21} \in \mathbb{R}^{(n-p_1)\times p_1}$ Then

$$\begin{bmatrix}0_{(n-n_0)\times n_0}\\ R_0\end{bmatrix}[E_1\;\; E_2] = \begin{bmatrix}E_{11} & 0_{p_1\times(n-p_1)}\\ E_{21} & I_{n-p_1}\end{bmatrix} \tag{7}$$

Equation (7) implies that

$$\operatorname{rank}\begin{bmatrix}I_n - \begin{bmatrix}0_{(n-n_0)\times n_0}\\ R_0\end{bmatrix}EP\\ CP\end{bmatrix} = \operatorname{rank}\begin{bmatrix}I_{p_1}-E_{11} & 0\\ -E_{21} & 0\\ S & 0\end{bmatrix} = p_1.$$

If we take $R = P\begin{bmatrix}0_{(n-n_0)\times n_0}\\ R_0\end{bmatrix}$ then

$$\operatorname{rank}\begin{bmatrix}I_n-RE\\ C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}P^{-1} & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}I_n - P\begin{bmatrix}0\\ R_0\end{bmatrix}E\\ C\end{bmatrix}P$$

$$= \operatorname{rank}\begin{bmatrix}I_n - \begin{bmatrix}0\\ R_0\end{bmatrix}EP\\ CP\end{bmatrix} = p_1.$$

Thus the conclusion (5) holds.

Algorithm to find the matrix R is given in the Appendix of this paper.

Remark 1. Above Theorem proves that if rectangular descriptor system (2) satisfies equation (3), then there exists following square restricted system equivalent

$$RE\dot{x} = RAx + RBu, \tag{8a}$$

$$y = Cx, \tag{8b}$$

which satisfies the equation (5). In the following Theorem, it is proved that in case of square system, converse of the Theorem 1 is true with R as an identity matrix.

Theorem 2. If descriptor system (1) is square, i.e. $m = n$, and satisfies the following condition:

$$\operatorname{rank}\begin{bmatrix}I_n-\tilde{E}\\ \tilde{C}\end{bmatrix} = \operatorname{rank}(\tilde{C}). \tag{9}$$

Then

$$\operatorname{rank}\begin{bmatrix}\tilde{E}\\ \tilde{C}\end{bmatrix} = n. \tag{10}$$

Proof. Let $\operatorname{rank}(\tilde{C}) = q$ then $\exists$ a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that $\tilde{C}P = [T\;\; 0]$. Where T is full column rank matrix.
Let $P^{-1}\tilde{E}P = \begin{bmatrix}\tilde{E}_{11} & \tilde{E}_{12}\\ \tilde{E}_{21} & \tilde{E}_{22}\end{bmatrix}$.
Thus

$$\operatorname{rank}\begin{bmatrix}I_n-\tilde{E}\\ C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}P^{-1} & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}I_n-\tilde{E}\\ C\end{bmatrix}P$$

$$= \operatorname{rank}\begin{bmatrix}I-\tilde{E}_{11} & \tilde{E}_{12}\\ \tilde{E}_{21} & I_{n-q}-\tilde{E}_{22}\\ T & 0\end{bmatrix} = q. \tag{11}$$

It follows from (11) that
$\tilde{E}_{12} = 0$ and $\tilde{E}_{22} = I_{n-q}$

$$\operatorname{rank}\begin{bmatrix}\tilde{E}\\ C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}P^{-1} & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}\tilde{E}\\ C\end{bmatrix}P \tag{12}$$

$$= \operatorname{rank}\begin{bmatrix}\tilde{E}_{11} & \tilde{E}_{12}\\ \tilde{E}_{21} & \tilde{E}_{22}\\ T & 0\end{bmatrix} \tag{13}$$

$$= \operatorname{rank}\begin{bmatrix}\tilde{E}_{11} & 0\\ \tilde{E}_{21} & I_{n-q}\\ T & 0\end{bmatrix} = n. \tag{14}$$

Hence the Theorem is proved.

Before proving the next Theorem, we shall define the observability of matrix pair (A,C) for some square matrix A. The matrix pair (A,C) is observable iff

$$\operatorname{rank}\left[C^T\;\; (CA)^T\;\; \ldots.\;\; (CA^{n-1})^T\right] = n,$$

which is equivalent to rank $\begin{bmatrix}\lambda I-A\\ C\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$ or $\lambda \in \sigma(A)$, where $\sigma(A) = \{\lambda|\lambda$ is a eigenvalue of A $\}$.

Theorem 3. Under the assumption of I-observability of the system (1), the following statements are equivalent.

(1) Descriptor system (1) is R-observable.
(2) Matrix pair (RA,C) is observable.

Proof. It is obvious that equation (5) implies the existence of $M \in \mathbb{R}^{n\times t}$ such that

$$RE = I - MC. \tag{15}$$

Thus for any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix}\lambda\tilde{E}-\tilde{A}\\ \tilde{C}\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}\lambda E-A\\ C\end{bmatrix} && (16)\\
&= \operatorname{rank}\begin{bmatrix}\lambda RE-RA\\ C\end{bmatrix} && (17)\\
&= \operatorname{rank}\begin{bmatrix}\lambda(I-MC)-RA\\ C\end{bmatrix}\\
&= \operatorname{rank}\begin{bmatrix}I & -\lambda M\\ 0 & I\end{bmatrix}\begin{bmatrix}\lambda I-RA\\ C\end{bmatrix}\\
&= \operatorname{rank}\begin{bmatrix}\lambda I-RA\\ C\end{bmatrix}.
\end{aligned}$$

Hence the Theorem is proved.

Remark 2. For descriptor system (1), in general, it is not easy to find particular λ such that condition $(H2)$ does not hold. Under assumption of impulse observability, in this paper, we have presented a method to find such λ by constructing a square equivalent system. This is explained in the Example 1.

4. CONCLUSION

In this paper we have given a method to check the R-observability of any irregular system under the assumption of I-observability. Application of this method can be found in observing the states of the given descriptor system, since I-observability and R-observability are sufficient conditions for designing an observer of irregular descriptor systems.

5. NUMERICAL EXAMPLES

Example 1. Consider the descriptor system (1) described by the following matrices:

$$\tilde{E} = \begin{bmatrix}1 & 2 & 4\\ 2 & 4 & 8\end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix}-1 & 5 & -5\\ -7 & 7 & -8\end{bmatrix}$$

$$\tilde{B} = [1\ \ 2]^T$$

$$\tilde{C} = [1\ \ 0\ \ 1]$$

First,we reduce this system in the form of system (2) by calculating

$$Q = \begin{bmatrix}-0.4472 & -0.8944\\ -0.8944 & 0.4472\end{bmatrix}$$

and

$$E = [-2.2361\ \ -4.4721\ \ -8.9443]$$
$$A = [6.7082\ \ -8.4971\ \ 9.3915]$$
$$B = [-2.2361]$$
$$C = \begin{bmatrix}-2.2361 & -1.3416 & 0.8944\\ 1.0000 & 0 & 1.0000\end{bmatrix}$$

Since $\operatorname{rank}\begin{bmatrix}E\\ C\end{bmatrix} = 3$, using the algorithm given in the Appendix, we calculate

$$R = \begin{bmatrix}0.0583\\ -0.1361\\ -0.0583\end{bmatrix}.$$

Then

$$RE = \begin{bmatrix}-0.1304 & -0.2609 & -0.5217\\ 0.3043 & 0.6087 & 1.2174\\ 0.1304 & 0.2609 & 0.5217\end{bmatrix}$$

$$RA = \begin{bmatrix}0.3913 & -0.4957 & 0.5478\\ -0.9130 & 1.1565 & -1.2783\\ -0.3913 & 0.4957 & -0.5478\end{bmatrix}$$

Now, we can check that $\operatorname{rank}\begin{bmatrix}I-RE\\ C\end{bmatrix} = 2$. Since $\operatorname{rank}\begin{bmatrix}I-RA\\ C\end{bmatrix} \neq 3$, matrix pair (RA, C) is not observable. Hence given system is also not R-observable and value of λ for which $(H2)$ is not satisfied is $\lambda = 1$.

Example 2. Consider (1) described by the following matrices:

$$\tilde{E} = \begin{bmatrix}0.5 & -2.5 & 0\\ 3 & -3 & 4\\ 2 & -1 & 3\end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix}-1 & 4.5 & -0.5\\ -7 & 7 & -8\\ -5 & 3 & -6\end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & -1\end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$$

First, we reduce this system in the form of system (2) by calculating

$$Q = \begin{bmatrix}-0.2163 & -0.8299 & -0.5143\\ 0.9029 & 0.0304 & -0.4287\\ 0.3714 & -0.5571 & 0.7428\end{bmatrix}$$

and

$$E = \begin{bmatrix}-3.6264 & 3.5447 & -4.8625\\ -0.3148 & -1.9197 & -1.1646\end{bmatrix}$$

$$A = \begin{bmatrix}8.5970 & -8.3254 & 9.8331\\ 1.0280 & 2.9897 & 1.8778\end{bmatrix}$$

$$B = \begin{bmatrix}-0.7306 & -0.8299 & -0.5319\\ 0.4742 & 0.0304 & 1.3620\end{bmatrix}$$

$$C = \begin{bmatrix}-0.1857 & 0 & -0.1857\\ 1.0000 & 0 & 1.0000\\ 1 & 0 & 0\end{bmatrix}.$$

Since $\operatorname{rank}\begin{bmatrix}E\\ C\end{bmatrix} = 3$, using the algorithm given in the Appendix, we calculate

$$R = \begin{bmatrix}0.4076 & 0.7526\\ 0.2181 & -0.1181\\ -0.6497 & -1.1997\end{bmatrix}.$$

Then

$$RE = \begin{bmatrix} -1.7151 & -0.0000 & -2.8585 \\ -0.7538 & 1.0000 & -0.9231 \\ 2.7339 & 0.0000 & 4.5565 \end{bmatrix}$$

$$RA = \begin{bmatrix} 4.2779 & -1.1434 & 5.4214 \\ 1.7538 & -2.1692 & 1.9231 \\ -6.8191 & 1.8226 & -8.6417 \end{bmatrix}.$$

Now, we can check that rank $\begin{bmatrix} I-RE \\ C \end{bmatrix} = 2$ and matrix pair (RA, C) is observable. Hence given descriptor system is R-observable.

REFERENCES

Berman, K. E., Campbell, S. L., and Petzold, L. R. (1996). *Numerical Solution of Initial Value Problems in Differential-Algebraic Equations.* SIAM.

Campbell, S. L. (1980). *Singular Systems of Differential Equations.* Research Notes in Mathematics. vol. 40.- London:Pitman.

Campbell, S. L. (1982). *Singular Systems of Differential Equations II.* Research Notes in Mathematics. vol. 61. London:Pitman.

Christodoulou, M. A. and Paraskevopoulos, P. N. (1985). Solvability, Controllability, and Observability of Singular Systems *J. Optimiz. Theory App.,* 45, 53-72.

Cobb, D. (1984). Controllability, observability, and duality in singular systems. *IEEE Trans. Automat. Contr.,* 29, 1076-1082.

Dai, L. (1989a). *Singular Control Systems.* Lecture Notes in Control and Information Sciences, Springer-Verlag.

Dai, L. (1989b). The difference between regularity and irregularity in singular systems. *Circuits Syst. Sig. Proc.,* 8(4), 435-444.

Darouach, M. and Boutayeb, M. (1995). Design of observers for descriptor systems. *IEEE Trans. Automat. Contr.,* 40(7), 1323-1327.

Duan, G.-R. (2010). *Analysis and Design of Descriptor Linear Systems.* Advances in Mechanics and Mathematics, 23, Springer.

Ishihara, J. Y. and Terra, M. H. (2001). Impulse Controllability and Observability of Rectangular Descriptor Systems. *IEEE Trans. Automat. Contr.,* 46, 991-994.

Lewis, F. L. (1986). A survey of linear singular systems. *Circuits Syst. Sig. Proc.,* 5(1), 3-36.

Luenberger, D. G. (1979). Nonlinear descriptor systems. *J. Eco. Dyn. Control,* 1, 219-242.

Rosenbrock, H. H. (1974). Structural Properties of Linear Dynamical Systems. *Int. J. Contr.,*. 20, 191-202.

Verghese, G. C., Levy, B. C., and Kailath, T. (1981). A Generalized State Space for Singular Systems. *IEEE Trans. Automat. Contr.,* 26, 811-831.

Yip, E. L., and Sincovec, R. F. (1981). Solvability, Controllability, and Observability of Continuous Descriptor Systems. *IEEE Trans. Automat. Contr.,* 26, 702-707.

Appendix A. ALGORITHM TO FIND THE MATRIX R:

1. Determine
$p_1 :=$ rank of matrix C
$n_0 \times n$:=order of matrix E.

2. Check rank $\begin{bmatrix} E \\ C \end{bmatrix} = n$, then go to steps 3-8.

3. Carry out the singular value decomposition (SVD) of matrix $C = U_1 \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T$.

4. Calculate $P = V_1 \begin{bmatrix} D_1^{-1} & 0 \\ 0 & I_{n-p_1} \end{bmatrix}$.

5. Calculate $E_2 = EP \begin{bmatrix} 0_{p_1 \times (n-p_1)} \\ I_{n-p_1} \end{bmatrix}$.

6. Carry out the SVD of matrix $E_2 = U_2 \begin{bmatrix} D_2 \\ 0 \end{bmatrix} V_2^T$.

7. Calculate $R_0 = \begin{bmatrix} 0 & I_{n_0+p_1-n} \\ V_2 D_2^{-1} & 0 \end{bmatrix} U_2^T$.

8. Calculate $R = P \begin{bmatrix} 0_{(n-n_0)\times n_0} \\ R_0 \end{bmatrix}$.