Asian Journal of Control, Vol. 16, No. 2, pp. 617–622, March 2014
Published online 8 April 2013 in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.704

–Brief Paper–

SQUARE-ROOT CUBATURE-QUADRATURE KALMAN FILTER

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ABSTRACT

In this paper, an on-going work introducing square-root extension of cubature-quadrature based Kalman filter is reported. The proposed method is named square-root cubature-quadrature Kalman filter (SR-CQKF). Unlike ordinary cubature-quadrature Kalman filter (CQKF), the proposed method propagates and updates square-root of the error covariance without performing Cholesky decomposition at each step. Moreover SR-CQKF ensures positive semi-definiteness of the state covariance matrix. With the help of two examples we show the superior performance of SR-CQKF compared to EKF and square root cubature Kalman filter.

Key Words: Kalman filter, Gauss quadrature rule, square-root filter, nonlinear estimation.

I. INTRODUCTION

Many researchers have devoted their best efforts to develop computationally affordable efficient algorithm for nonlinear estimation because it is essential in many real-life problems. Until recently, the extended Kalman filter (EKF) has been the natural choice of practitioners for solving nonlinear problems. The EKF uses first order linearization to approximately calculate the mean and covariance of non Gaussian probability density function. Due to such crude approximation, the filter looses track if the nonlinearity or uncertainty in the system is high. Limitations of EKF for severely nonlinear problems are well documented in earlier literature [1,2].

To alleviate the problem, in [3] we proposed a technique named cubature-quadrature Kalman filter (CQKF). CQKF is based on spherical cubature and Gauss-Laguerre quadrature rule of numerical integration. The proposed technique is a more generalized form of cubature Kalman filter (CKF) [4], (which we here call CQKF-1) and under single Gauss-Laguerre quadrature point approximation it merges with CKF.

In CQKF, it is necessary to perform Cholesky decomposition at each step. Due to accumulated round off error associated with processing software, Cholesky decomposition sometimes leads to negative definite covariance matrices, causing the filter to stop. Unscented Kalman filter (UKF) [5] and Gauss Hermite filter (GHF) also suffer from similar type of hitch which has been rectified by Merwe *et al*. [6], and Arasaratnam *et al*. [7], by proposing square-root formulation. In [7], Arasaratnam and Heykin proposed a square root version of the Gauss-Hermite filter algorithm which uses the Gauss-Hermite quadrature method to evaluate intractable integrals. In the present paper, we propose the square root version of cubature quadrature filter which uses third order cubature and Gauss-Laguerre quadrature rule to evaluate the integrals. The methodology adopted here to estimate the states of a nonlinear system is totally different from [7], whereas the square root method we formulate is similar to [6] and [7].

The square-root version of UKF is now being applied to practical problems [8,9] for enhanced computational stability. Motivated by earlier works [6,7] and their usefulness, we propose the square-root version of CQKF (SR-CQKF) which enhances numerical stability by ensuring positive semi-definiteness of the error covariance matrix. Unlike the CQKF, where Cholesky decomposition, computationally the most expensive operation, has to be performed at each step, the SR-CQKF propagates the state estimate and its square-root covariance. With the help of the examples, we show that SR-CQKF outperforms EKF and SR-CKF.

The rest of the paper is organized as follows. The next section provides the brief description of CQKF algorithm. This is followed by elaborate algorithm of SR-CQKF. Finally simulation results, and concluding remarks are given in Section 4 and 5 respectively.

II. CUBATURE-QUADRATURE KALMAN FILTER

We developed a novel algorithm [3] where the multivariate moment integral has been computed numerically using cubature rule and multiple Gauss-Laguerre quadrature points. The algorithm has been named the cubature-quadrature Kalman filter. In this section, we briefly describe the algorithm. For detailed derivation readers are referred to [3].

Let us consider a nonlinear plant described by the state and measurement equations as follows:

$$x_{k+1} = \phi(x_k) + \eta_k \tag{1}$$

$$y_k = \gamma(x_k) + v_k \tag{2}$$

where $x_k \in \mathbb{R}^n$ denotes the state of the system, $y_k \in \mathbb{R}^p$ is the measurement at the instant k where $k = \{0, 1, 2, 3, \ldots N\}$, $\phi(x_k)$

Manuscript received June 3, 2012; revised October 5, 2012; accepted November 14, 2012.
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and $\gamma(x_k)$ are known nonlinear functions of x_k and k. The process noise $\eta_k \in \mathbb{R}^n$ and measurement noise $v_k \in \mathbb{R}^p$ are assumed to be uncorrelated and normally distributed with covariance Q_k and R_k respectively.

CQKF approximately evaluates the intractable integrals encountered in Baysian filtering framework using cubature-quadrature points and corresponding weights associated with them. For n dimensional problem solved with third order spherical cubature and n' order of quadrature rule, total $2nn'$ number of points and associated weights need to be calculated.

The algorithm of the cubature-quadrature Kalman filter (CQKF) could be summarized as follows:

Step i. Filter initialization.

- Initialize the filter with $\hat{x}_{0|0}$ and $P_{0|0}$.
- Calculate the cubature-quadrature (CQ) points ξ_j and their corresponding weights $w_j(j = 1, 2, \ldots, 2nn')$.

Step ii. Predictor step.

- Perform Cholesky decomposition of posterior error covariance

$$P_{k|k} = S_{k|k} S_{k|k}^T \tag{3}$$

- Evaluate cubature-quadrature points

$$\chi_{j,k|k} = S_{k|k}\xi_j + \hat{x}_{k|k} \tag{4}$$

- Update cubature-quadrature points

$$\chi_{j,k+1|k} = \phi(\chi_{j,k|k}) \tag{5}$$

- Compute time updated mean and covariance

$$\hat{x}_{k+1|k} = \sum_{j=1}^{2nn'} w_j \chi_{j,k+1|k} \tag{6}$$

$$P_{k+1|k} = \sum_{j=1}^{2nn'} w_j \left[\chi_{j,k+1|k} - \hat{x}_{k+1|k}\right]\left[\chi_{j,k+1|k} - \hat{x}_{k+1|k}\right]^T + Q_k \tag{7}$$

Step iii. Corrector step or measurement update.

- Perform Cholesky decomposition of prior error covariance

$$P_{k+1|k} = S_{k+1|k} S_{k+1|k}^T \tag{8}$$

- Evaluate cubature-quadrature points

$$\chi_{j,k+1|k} = S_{k+1|k}\xi_j + \hat{x}_{k+1|k} \tag{9}$$

where $j = 1,2, \ldots 2nn'$.

- Find the predicted measurements at each cubature-quadrature points

$$Y_{j,k+1|k} = \gamma(\chi_{j,k+1|k}) \tag{10}$$

- Estimate the predicted measurement

$$\hat{y}_{k+1} = \sum_{j=1}^{2nn'} w_j Y_{j,k+1|k} \tag{11}$$

- Calculate the covariances

$$P_{y_{k+1}y_{k+1}} = \sum_{j=1}^{2nn'} w_j \left[Y_{j,k+1|k} - \hat{y}_{k+1}\right]\left[Y_{j,k+1|k} - \hat{y}_{k+1}\right]^T + R_k \tag{12}$$

$$P_{x_{k+1}y_{k+1}} = \sum_{j=1}^{2nn'} w_j \left[\chi_{j,k+1|k} - \hat{x}_{k+1|k}\right]\left[Y_{j,k+1|k} - \hat{y}_{k+1}\right]^T \tag{13}$$

- Calculate Kalman gain

$$K_{k+1} = P_{x_{k+1}y_{k+1}} P_{y_{k+1}y_{k+1}}^{-1} \tag{14}$$

- Compute posterior state values

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - \hat{y}_{k+1}\right) \tag{15}$$

- Posterior error covariance matrix is given by

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1} P_{y_{k+1}y_{k+1}} K_{k+1}^T \tag{16}$$

III. SQUARE-ROOT CUBATURE-QUADRATURE KALMAN FILTER

The SR-CQKF is an algebraic re-formulation of CQKF algorithm, where Cholesky decomposition need not to be performed at each step. Due to limited word length in processing software, round off error accumulates and after some iterations the positive semi-definite nature of the error covariance matrix may be lost. To circumvent the problem, in this section, we re-formulate the CQKF algorithm based on orthogonal-triangular decomposition, popularly known as QR decomposition.

The error covariance matrix, P can be factorized as,

$$P = AA^T \tag{17}$$

where $P \in \mathbb{R}^{n\times n}$, and $A \in \mathbb{R}^{n\times m}$ with $m \geq n$. Due to increased dimension of A, Arasaratnam and Haykin [7] called it as *"fat"* matrix. The QR decomposition of matrix A^T is given by, $A^T = QR$ where $Q \in \mathbb{R}^{m\times m}$ is *orthogonal*, and $R \in \mathbb{R}^{m\times n}$ is *upper triangular*. With the QR decomposition described above, the error covariance matrix can be written as

$$P = AA^T = R^T Q^T QR = \tilde{R}^T \tilde{R} = SS^T \tag{18}$$

$\tilde{R}$ is the upper triangular part of R matrix, $\tilde{R} \in \mathbb{R}^{n\times n}$, where $\tilde{R}^T = S$.

The detailed algorithm of square-root cubature-quadrature Kalman filter is provided below. We use the notation $qr\{\}$ to denote the QR decomposition of a matrix and $uptri\{\}$ to select the upper triangular part of a matrix.

Step i. Filter initialization.

- Initialize the filter with $\hat{x}_{0|0}$ and $P_{0|0} = S_{0|0}S_{0|0}^T$
- Calculate the cubature-quadrature (CQ) points ξ_j and their corresponding weights w_j ($j = 1, 2, \ldots, 2nn'$). The methodology to calculate n' order cubature quadrature points and associated weights has been described in paper [3].

Step ii. Predictor step.

- Evaluate cubature-quadrature points

$$\chi_{j,k|k} = S_{k|k}\xi_j + \hat{x}_{k|k} \tag{19}$$

- Update cubature-quadrature points

$$\chi_{j,k+1|k} = \phi(\chi_{j,k|k}) \tag{20}$$

- Compute time updated mean

$$\hat{x}_{k+1|k} = \sum_{j=1}^{2nn'} w_j \chi_{j,k+1|k} \tag{21}$$

- Evaluate square-root weight matrix

$$W = \begin{bmatrix} \sqrt{w_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{w_{2nn'}} \end{bmatrix} \tag{22}$$

- Calculate the matrix of square-root weighted cubature-quadrature points after prior mean subtraction

$$\chi^*_{k+1|k} = [\chi_{1,k+1|k} - \hat{x}_{k+1|k} \quad \chi_{2,k+1|k} - \hat{x}_{k+1|k} \quad \cdots \quad \chi_{2nn',k+1|k} - \hat{x}_{k+1|k}]W \tag{23}$$

- Perform QR decomposition

$$\Re_{k+1|k} = qr\left\{\left[\chi^*_{k+1|k} \quad \sqrt{Q_k}\right]\right\} \tag{24}$$

- Estimate the square-root of predicted error covariance

$$S_{k+1|k} = uptri\{\Re_{k+1|k}\} \tag{25}$$

Step iii. Measurement update.

- Evaluate cubature-quadrature points

$$\chi_{j,k+1|k} = S_{k+1|k}\xi_j + \hat{x}_{k+1|k} \tag{26}$$

- Find the predicted measurements at each CQ points

$$Y_{j,k+1|k} = \gamma(\chi_{j,k+1|k}) \tag{27}$$

- Estimate the predicted measurement

$$\hat{y}_{k+1} = \sum_{j=1}^{2nn'} w_j Y_{j,k+1|k} \tag{28}$$

- Calculate the square-root weighted measurement matrix

$$Y^*_{k+1|k} = [Y_{1,k+1|k} - \hat{y}_{k+1|k} \quad Y_{2,k+1|k} - \hat{y}_{k+1|k} \quad \cdots \quad Y_{2nn',k+1|k} - \hat{y}_{k+1|k}]W \tag{29}$$

- Perform QR decomposition

$$\Re_{y_{k+1}y_{k+1}} = qr\left\{\left[Y^*_{k+1|k} \quad \sqrt{R}_k\right]\right\} \tag{30}$$

- Calculate the square root of innovation covariance

$$S_{y_{k+1}y_{k+1}} = uptri\{\Re_{y_{k+1}y_{k+1}}\} \tag{31}$$

- Calculate the cross covariance matrix

$$P_{x_{k+1}y_{k+1}} = \chi^*_{k+1|k}Y^*_{k+1|k} \tag{32}$$

- Calculate Kalman gain

$$K = \left(P_{x_{k+1}y_{k+1}} / S^T_{y_{k+1}y_{k+1}}\right) / S_{y_{k+1}y_{k+1}} \tag{33}$$

- Compute posterior state values

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K(y_{k+1} - \hat{y}_{k+1}) \tag{34}$$

- Calculate QR decomposition

$$\Re_{k+1|k+1} = qr\left\{\left[\chi^*_{k+1|k} - KY^*_{k+1|k} \quad \sqrt{Q}_k \quad K\sqrt{R}_k\right]\right\} \tag{35}$$

- Calculate square root of posterior error covariance matrix

$$S_{k+1k+1} = uptri\{\Re_{k+1|k+1}\} \tag{36}$$

Note 1. The equation (35) can be derived from equations (14), (16), (24), (30), and (32) by matrix manipulation. For detailed steps readers are refered to [7].

Note 2. The practitioners who want to implement the algorithm in MATLAB software environment, should note that the command *chol* (P) returns the matrix S^T (where $P = SS^T$) rather than S. Accordingly the expressions for next steps need to be evaluated.

IV. SIMULATION RESULTS

In this section, the formulated algorithm described above has been applied to solve a single dimensional as well as multi dimensional nonlinear problem.

Example 1. Single dimensional process.

The plant model, inspired by [10], has one unstable and two stable equilibrium points and given by

$$x_{k+1} = \phi(x_k) + \eta_k$$

where,

$$\phi(x) = x + \Delta t 5x(1 - x^2), \quad \eta_k \sim \aleph(0, b^2\Delta t)$$

The measurement model is

$$y_k = \gamma(x_k) + v_k$$

where,

$$\gamma(x) = \Delta t x(1 - 0.5x), \quad v_k \sim \aleph(0, d^2\Delta t)$$

and the value of $\Delta t = 0.01$ seconds, $x_0 = -0.2$, $\hat{x}_{0|0} = 0.8$, $P_{0|0} = 2$, $b = 0.5$ and $d = 0.1$. We have considered the time span from 0 to 4 seconds. The system has three equilibrium points, of which, the one at the origin is unstable and the other two are at $\pm$ 1 and stable. In the absence of any bias, the system settles around either of the two stable equilibrium points. The problem becomes challenging because moderate estimation error forces the estimate to settle down at the wrong equilibrium point, leading to a track loss situation.

We solve the estimation problem using EKF and SR-CQKF. The percentage fail counts for different filters are shown in Table I. The percentage fail count is defined as the number of cases where estimation error at 4th second is more than unity out of 100 Monte Carlo runs. In our notation, CQKF-x indicates an x order Gauss-Laguerre cubature-quadrature filter. Under the first order Gauss-Laguerre approximation CQKF reduces to CKF. The numbers mentioned in the table show the improvement of performance with the square-root CQKF.

Table I. Percentage fail count.

Filter	Fail Count
EKF	22%
SR-CKF or SR-CQKF-1	4%
SR-CQKF-2	1.05%
SR-CQKF-3	1.02%
SR-CQKF-4	1.02%

Fig. 1 shows the root mean square error (RMSE) obtained from SR-CQKF and EKF out of 10,000 Monte Carlo runs. It can be seen from Fig. 1, that RMSE of higher order SR-CQKF converges more quickly than first order SR-CQKF or SR-CKF. However no significant improvement of performance is noticed with more than second order SR-CQKF.

Example 2. Lorenz system.

Inspired from earlier works [10,11], in this example, we consider the discrete time Lorenz attractor, one of the classic icons in nonlinear dynamics. The attractor is available in the literature from 1963, mainly due to the work of meteorologist Edward Lorenz. Although a higher dimensional Lorenz attractor may exists [12,13] in real life problems, here we consider three dimensional system. The process equation is given by

$$x_{k+1} = \phi(x_k) + b\eta_k$$

where,

$$\phi(x) = x + \Delta t f(x), \quad \eta_k \sim \aleph(0, \Delta t)$$

The measurement equation is expressed as

$$y_k = \gamma(x_k) + dv_k$$

where,

$$\gamma(x) = \Delta t h(x), \quad v_k \sim \aleph(0, \Delta t)$$

Here $x_k = [x_{1,k}\ x_{2,k}\ x_{3,k}] \in \mathbb{R}^3$ is state variable. The parameters $b \in \mathbb{R}^3$, and $d \in \mathbb{R}$ are constant whose values are taken as $b = [0\ 0\ 0.5]^T$ and $d = 0.065$. The functions $f(x)$ and $h(x)$ are given by

$$f(x) = [\alpha(-x_1 + x_2) \quad \beta x_1 - x_2 - x_1 x_3 \quad -\gamma x_3 + x_1 x_2]^T$$

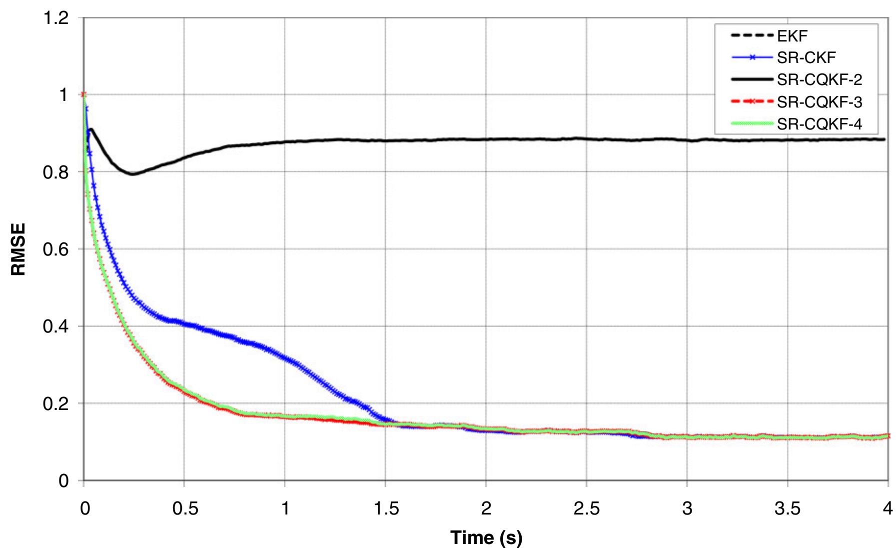

Fig. 1. RMSE plot of SR-CQKF with higher order Gauss-Laguerre quadrature points.

Table II. Average root mean square error over time span.

Filter	Averaged RMSE (State 1)	Averaged RMSE (State 2)	Averaged RMSE (State 3)
EKF	10.0134	11.9914	0.18
SR-CKF or SR-CQKF-1	5.6421	6.7814	0.18
SR-CQKF-2	4.1049	5.0641	0.18
SR-CQKF-3	4.1010	5.0594	0.18
SR-CQKF-4	4.0967	5.0543	0.18

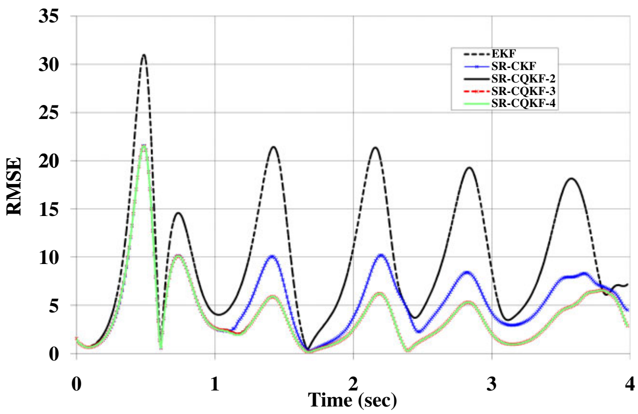

Fig. 2. RMSE plot of EKF and SR-CQKF for first state variable.

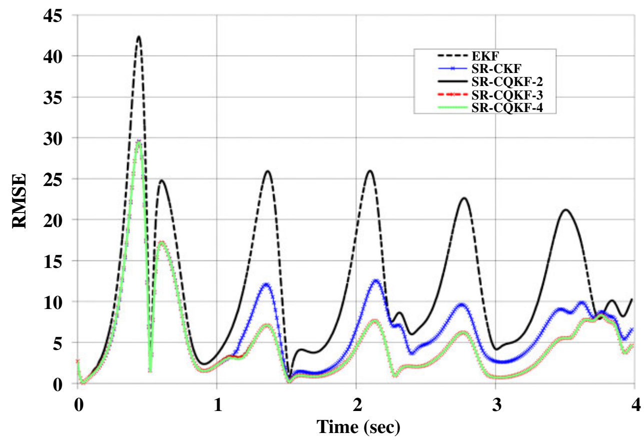

Fig. 3. RMSE plot of EKF and SR-CQKF for second state variable.

and

$$h(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

The three parameters, α (called the Prandtl number), β (called the Rayleigh number), and γ have great impact on the system. The system has three unstable equilibrium points and for $\alpha \neq 0$, and $\gamma(\beta-1) \geq 0$ they are at $[0 \quad 0 \quad 0]^T$, $\left[\sqrt{\gamma(\beta-1)} \quad \sqrt{\gamma(\beta-1)} \quad (\beta-1)\right]^T$, and $\left[-\sqrt{\gamma(\beta-1)} \quad -\sqrt{\gamma(\beta-1)} \quad (\beta-1)\right]^T$. We chose the system with classical parameter values: $\alpha = 10$, $\gamma = 8/3$, and $\beta = 28$, for which almost all points in phase space tend to a strange attractor [13].

The initial truth value of state is taken as $x_0 = [-0.2 \ -0.3 \ -0.5]^T$. We simulate during the time interval of 0 to 4 seconds with the sampling time $\Delta t = 0.01$ seconds. The initial posterior estimate is assumed as $\hat{x}_{0|0} = [1.35 \quad -3 \quad 6]^T$ with initial error covariance $P_{0|0} = 0.35 I_3$.

The above described problem has been solved using square-root version of CQKF with higher order of Gauss-Laguerre approximation. We observe that sometimes ordinary CQKF fails to run due to non positive definite error covariance matrix. The same problem does not occur during SR-CQKF as expected from the formulation. The root mean square error, averaged over time span, obtained from 50 Monte Carlo runs, has been tabulated in Table II. In Figs 2–4, we show the root mean square errors (RMSEs) of the states obtained from EKF and SR-CQKF. RMSE was calculated from 50 Monte Carlo runs. The results obtained from the simulation run may be summarized as follows:

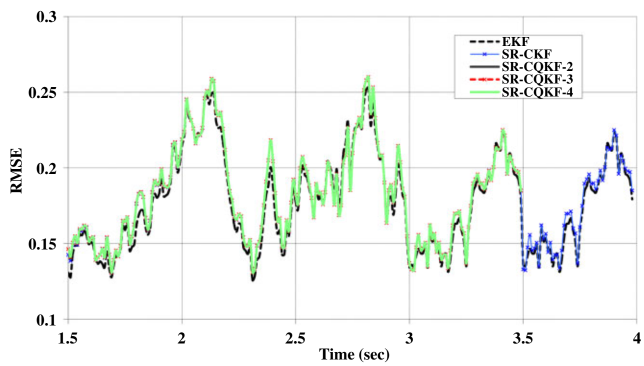

Fig. 4. RMSE plot of EKF and SR-CQKF for third state variable.

- From Fig. 4, no improvement in estimation of third state variable has been observed with SR-CQKF in comparison with EKF. This is also supported from the numbers displayed in Table II.
- EKF uses first order linearization to approximately calculate the mean and covariance of non Gaussian probability density function. Cubature quadrature Kalman filter uses cubature rule and Gauss-Laguerre quadrature points to

determine the mean and covariance of posterior probability density function. The moment calculation is more accurate in the second case; hence improvement in performance is expected compared to EKF. The argument is well supported by Figs 2–3, where we notice that for first and second state variable, SR-CQKF-1 or SR-CKF performs significantly better than EKF.

- Further for the first and second state variable, SR-CQKF-2 performs considerably better than SR-CQKF-1. Although from the Figs 2–3 no noticeable improvement is observed with SR-CQKF with the order higher than 2.
- From Table II, it is clear that SR-CQKF provides significantly lower averaged RMSE in comparison to EKF. Also the performance of the SR-CQKF improves with the higher order Gauss-Laguerre quadrature approximation. However after the 4th order, SR-CQKF does not offer any improvement, hence it is not displayed in the table.
- Computational times of SR-CQKF and CQKF have been compared. It has been observed that CQKF-1 and SR-CQKF-1 take 7.83 seconds and 4.19 seconds respectively for 50 MC runs in MATLAB software in a personal computer with a 2.40 GHz processor. The result indicates lower computational cost of square root cubature quadrature Kalman filter compared to ordinary CQKF.

V. DISCUSSIONS AND CONCLUSION

In this paper, the square-root extension of cubature-quadrature Kalman filter has been proposed. In comparison to CQKF, the proposed estimator has better numerical properties and guaranteed positive semi-definiteness of error covariance matrix. Instead of performing Cholesky decomposition at each step, the proposed filter propagates and updates square-root of the error covariance. With the help of examples, the superiority of the proposed algorithm over SR-CKF and EKF is demonstrated. The simulation results also reveal the performance improvement of SR-CQKF with higher order Gauss-Laguerre quadrature approximation.

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